DISCRETE LAGRANGIAN AND HAMILTONIAN MECHANICS ON LIE GROUPOIDS

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ABSTRACT. The purpose of this paper is to describe geometrically discrete Lagrangian and Hamiltonian Mechanics on Lie groupoids. From a variational principle we derive the discrete Euler-Lagrange equations and we introduce a symplectic 2-section, which is preserved by the Lagrange evolution operator. In terms of the discrete Legendre transformations we define the Hamiltonian evolution operator which is a symplectic map with respect to the canonical symplectic 2-section on the prolongation of the dual of the Lie algebroid of the given groupoid. The equations we get include as particular cases the classical discrete Euler-Lagrange equations, the discrete Euler-Poincaré and discrete Lagrange-Poincaré equations. Our results can be important for the construction of geometric integrators for continuous Lagrangian systems.

CONTENTS

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1. Introduction

During the last decade, much effort has been devoted to construction of geometric integrators for Lagrangian systems using a discrete variational principle (see [21] and references therein). In particular, this effort has been concentrated for the case of discrete Lagrangian functions L on the cartesian product $Q\times Q$ of a differentiable manifold. This cartesian product plays the role of a "discretized version" of the standard velocity phase space TQ. Applying a natural discrete variational principle, one obtains a second order recursion operator $\xi : Q \times Q \longrightarrow Q \times Q$ assigning to each input pair (x, y) the output pair (y, z). When the discrete Lagrangian is an approximation of a continuous Lagrangian function (more appropriately, when the discrete Lagrangian approximates the integral action for L) we obtain a numerical integrator which inherits some of the geometric properties of the continuous Lagrangian (symplecticity, momentum preservation). Although this type of geometric integrators have been mainly considered for conservative systems, the extension to geometric integrators for more involved situations is relatively easy, since, in some sense, many of the constructions mimic the corresponding ones for the continuous counterpart. In this sense, it has been recently shown how discrete variational mechanics can include forced or dissipative systems, holonomic constraints, explicitely time-dependent systems, frictional contact, nonholonomic constraints, multisymplectic fields theories... All these geometric integrators have demonstrated, in worked examples, an exceptionally good longtime behavior and obviously this research is of great interest for numerical and geometric considerations (see [8]).

On the other hand, Moser and Veselov [26] consider also discrete Lagrangian systems evolving on a Lie group. All this examples leads to A. Weinstein [31] to study discrete mechanics on Lie groupoids, which is a structure that includes as particular examples the case of cartesian products $Q \times Q$ as well as Lie groups.

A Lie groupoid G is a natural generalization of the concept of a Lie group, where now not all elements are composable. The product $g_1 g_2$ of two elements is only defined on the set of composable pairs $G_2 = \{(g, h) \in G \times G \mid \beta(g) = \alpha(h)\}$ where $\alpha : G \longrightarrow M$ and $\beta : G \longrightarrow M$ are the source and target maps over a base manifold M. This concept was introduced in differential geometry by Ch. Ereshmann in the 1950's. The infinitesimal version of a Lie groupoid G is the Lie algebroid $AG \longrightarrow M$, which is the restriction of the vertical bundle of α to the submanifold of the identities.

We may thought a Lie algebroid A over a manifold M, with projection $\tau : A \to M$, as a generalized version of the tangent bundle to M. The geometry and dynamics on Lie algebroids have been extensively studied during the past years. In particular, one of the authors of this paper (see [22]) developed a geometric formalism of mechanics on Lie algebroids similar to Klein's formalism [11] of the ordinary Lagrangian mechanics and more recently a description of the Hamiltonian dynamics on a Lie algebroid was given in [14, 23] (see also [28]).

The key concept in this theory is the prolongation, $\mathcal{P}^\tau A$, of the Lie algebroid over the fibred projection τ (for the Lagrangian formalism) and the prolongation, $\mathcal{P}^{\tau^*} A$, over the dual fibred projection $\tau^* : A^* \longrightarrow M$ (for the Hamiltonian formalism). See [14] for more details. Of course, when the Lie algebroid is $A = TQ$ we obtain that $\mathcal{P}^\tau A = T(TQ)$ and $\mathcal{P}^{\tau^*} A = T(T^*Q)$, recovering the classical case. An alternative approach, using the linear Poisson structure on A^* and the canonical isomorphism between T^*A and T^*A^* was discussed in [7].

Taking as starting point the results by A. Weinstein [31], we elucidate in this paper the geometry of Lagrangian systems on Lie groupoids and its Hamiltonian counterpart. Weinstein gave a variational derivation of the discrete Euler-Lagrange equations for a Lagrangian $L : G \to \mathbb{R}$ on a Lie groupoid G. We show that the appropriate space to develop a geometric formalism for these equations is the Lie algebroid $\mathcal{P}^\tau G \equiv V\beta \oplus_G V\alpha \to G$ (see section 3 for the definition of the Lie algebroid structure). Note that $\mathcal{P}^\tau G$ is the total space of the prolongation of the Lie groupoid G over the vector bundle projection $\tau : AG \to M$, and that the Lie algebroid of $\mathcal{P}^\tau G$ is just the prolongation $\mathcal{P}^\tau(AG)$ (the space were the continuous Lagrangian Mechanics is developed). Using the Lie algebroid structure of $\mathcal{P}^\tau G$ we may describe discrete Mechanics on the Lie groupoid G. In particular,

- We give a variational derivation of the discrete Euler-Lagrange equations:
$$\overleftarrow{X}(g)(L) - \overrightarrow{X}(h)(L) = 0$$
for every section X of AG, where the right or left arrow denotes the induced right and left-invariant vector field on G.
- We introduce two Poincaré-Cartan 1-sections Θ_L^+ and Θ_L^-, and an unique Poincaré-Cartan 2-section, Ω_L, on the Lie algebroid $P^\tau G \longrightarrow G$.
- We study the discrete Lagrangian evolution operator $\xi : G \longrightarrow G$ and its preservation properties. In particular, we prove that $(\mathcal{P}^\tau\xi, \xi)^*\Omega_L = \Omega_L$, where $\mathcal{P}^\tau\xi$ is the natural prolongation of ξ to $\mathcal{P}^\tau G$.
- Reduction theory is stablished in terms of morphisms of Lie groupoids.
- The associated Hamiltonian formalism is developed using the discrete Legendre transformations $\mathbb{F}^+L : G \to A^*G$ and $\mathbb{F}^-L : G \to A^*G$.
- A complete characterization of the regularity of a Lagrangian on a Lie groupoid is given in terms of the symplecticity of Ω_L or, alternatively, in terms of the regularity of the discrete Legendre transformations. In particular, Theorem 4.13 solves the question posed by Weinstein [31] about the regularity conditions for a discrete Lagrangian function on more general Lie groupoids than the cartesian product $Q \times Q$. In the regular case, we define the Hamiltonian evolution operator and we prove that it defines a symplectic map.
- We prove a Noether's theorem for discrete Mechanics on Lie groupoids.
- Finally, some illustrative examples are shown, for instance, discrete Mechanics on the cartesian product $Q\times Q$, on Lie groups (discrete Lie-Poisson equations), on action Lie groupoids (discrete Euler-Poincaré equations) and on gauge or Atiyah Lie groupoids (discrete Lagrange-Poincaré equations).

We expect that the results of this paper could be relevant in the construction of new geometric integrators, in particular, for the numerical integration of dynamical systems with symmetry.

The paper is structured as follows. In Section 2 we review some basic results on Lie algebroids and Lie groupoids. Section 3 is devoted to study the Lie algebroid structure of the vector bundle $\mathcal{P}^\tau G \equiv V\beta \oplus_G V\alpha \ \to G$. The main results of the paper appear in Section 4, where the geometric structure of discrete Mechanics on Lie groupoids is given. Finally, in Section 5, we study several examples of the theory.

2. Lie algebroids and Lie groupoids

2.1. **Lie algebroids.** A ***Lie algebroid*** A over a manifold M is a real vector bundle $\tau : A \to M$ together with a Lie bracket $[\![\cdot,\cdot]\!]$ on the space $\Gamma(\tau)$ of the global cross sections of $\tau : A \to M$ and a bundle map $\rho : A \to TM$, called ***the anchor map***, such that if we also denote by $\rho : \Gamma(\tau) \to \mathfrak{X}(M)$ the homomorphism of $C^\infty(M)$-modules induced by the anchor map then
$$[\![X, fY]\!] = f[\![X, Y]\!] + \rho(X)(f)Y, \tag{2.1}$$

for $X, Y \in \Gamma(\tau)$ and $f \in C^\infty(M)$ (see [17]).

If $X, Y, Z \in \Gamma(\tau)$ and $f \in C^\infty(M)$ then, using (2.1) and the fact that $[\![\cdot,\cdot]\!]$ is a Lie bracket, we obtain that

$$[\![[\![X,Y]\!], fZ]\!] = f([\![X,[\![Y,Z]\!]]\!] - [\![Y,[\![X,Z]\!]]\!]) + [\rho(X),\rho(Y)](f)Z. \tag{2.2}$$

On the other hand, from (2.1), it follows that

$$[\![[\![X,Y]\!], fZ]\!] = f[\![[\![X,Y]\!], Z]\!] + \rho[\![X,Y]\!](f)Z. \tag{2.3}$$

Thus, using (2.2), (2.3) and the fact that $[\![\cdot,\cdot]\!]$ is a Lie bracket, we conclude that

$$\rho[\![X,Y]\!] = [\rho(X),\rho(Y)],$$

that is, $\rho : \Gamma(\tau) \to \mathfrak{X}(M)$ is a homomorphism between the Lie algebras $(\Gamma(\tau), [\![\cdot,\cdot]\!])$ and $(\mathfrak{X}(M), [\cdot,\cdot])$.

If $(A, [\![\cdot,\cdot]\!], \rho)$ is a Lie algebroid over M, one may define ***the differential of*** A, $d : \Gamma(\wedge^k \tau^*) \to \Gamma(\wedge^{k+1}\tau^*)$, as follows

$$\begin{aligned} d\mu(X_0,\ldots,X_k) &= \sum_{i=0}^{k}(-1)^i \rho(X_i)(\mu(X_0,\ldots,\widehat{X_i},\ldots,X_k)) \\ &\quad + \sum_{i<j}(-1)^{i+j}\mu([\![X_i,X_j]\!], X_0,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_k), \end{aligned} \tag{2.4}$$

for $\mu \in \Gamma(\wedge^k\tau^*)$ and $X_0,\ldots,X_k \in \Gamma(\tau)$. d is a cohomology operator, that is, $d^2 = 0$. In particular, if $f : M \longrightarrow \mathbb{R}$ is a real smooth function then $df(X) = \rho(X)f$, for $X \in \Gamma(\tau)$. We may also define the Lie derivative with respect to a section X of A as the operator $\mathcal{L}_X : \Gamma(\Lambda^k A^*) \longrightarrow \Gamma(\Lambda^k A^*)$ given by $\mathcal{L}_X = i_X \circ d + d \circ i_X$ (for more details, see [17]).

Trivial examples of Lie algebroids are a real Lie algebra $\mathfrak{g}$ of finite dimension (in this case, the base space is a single point) and the tangent bundle TM of a manifold M. Other examples of Lie algebroids are: i) the vertical bundle $(\tau_P)_{|V\pi} : V\pi \to P$ of a fibration $\pi : P \to M$ (and, in general, the tangent vectors to a foliation of finite dimension on a manifold P); ii) ***the Atiyah algebroid associated with a principal G-bundle*** (see [14, 17]); iii) ***the prolongation $\mathcal{P}^\pi A$ of a Lie algebroid A over a fibration*** $\pi : P \to M$ (see [9, 14]) and iv) ***the action Lie algebroid*** $A \ltimes f$ over a map $f : M' \to M$ (see [9, 14]).

Now, let $(A, [\![\cdot,\cdot]\!], \rho)$ (resp., $(A', [\![\cdot,\cdot]\!]', \rho')$) be a Lie algebroid over a manifold M (resp., M') and suppose that $\Psi : A \to A'$ is a vector bundle morphism over the map $\Psi_0 : M \to M'$. Then, the pair (Ψ, Ψ_0) is said to be a ***Lie algebroid morphism*** if

$$d((\Psi,\Psi_0)^*\phi') = (\Psi,\Psi_0)^*(d'\phi'), \quad \text{for all } \phi' \in \Gamma(\wedge^k(A')^*) \text{ and for all } k, \tag{2.5}$$

where d (resp., d') is the differential of the Lie algebroid A (resp., A') (see [14]). In the particular case when $M = M'$ and $\Psi_0 = Id$ then (2.5) holds if and only if

$$[\![\Psi\circ X, \Psi\circ Y]\!]' = \Psi[\![X,Y]\!], \quad \rho'(\Psi X) = \rho(X), \quad \text{for } X, Y \in \Gamma(\tau).$$

2.2. **Lie groupoids.** In this Section, we will recall the definition of a Lie groupoid and some generalities about them are explained (for more details, see [3, 17]).

A ***groupoid*** over a set M is a set G together with the following structural maps:

- A pair of maps $\alpha : G \to M$, the ***source***, and $\beta : G \to M$, the ***target***. Thus, an element $g \in G$ is thought as an arrow from $x = \alpha(g)$ to $y = \beta(g)$ in M

$$x = \alpha(g) \xrightarrow{\quad g \quad} y = \beta(g)$$

  The maps α and β define the set of composable pairs

$$G_2 = \{(g,h) \in G \times G / \beta(g) = \alpha(h)\}.$$

- A ***multiplication*** $m : G_2 \to G$, to be denoted simply by $m(g,h) = gh$, such that
  - $\alpha(gh) = \alpha(g)$ and $\beta(gh) = \beta(h)$.
  - $g(hk) = (gh)k$.

  If g is an arrow from $x = \alpha(g)$ to $y = \beta(g) = \alpha(h)$ and h is an arrow from y to $z = \beta(h)$ then gh is the composite arrow from x to z

$$\begin{tikzcd} x = \alpha(g) = \alpha(gh) \arrow[r, bend left, "g"] \arrow[rr, bend left=40, "gh"] & y = \beta(g) = \alpha(h) \arrow[r, bend left, "h"] & z = \beta(h) = \beta(gh) \end{tikzcd}$$

- An ***identity section*** $\epsilon : M \to G$ such that
  - $\epsilon(\alpha(g))g = g$ and $g\epsilon(\beta(g)) = g$.
- An ***inversion map*** $i : G \to G$, to be denoted simply by $i(g) = g^{-1}$, such that
  - $g^{-1}g = \epsilon(\beta(g))$ and $gg^{-1} = \epsilon(\alpha(g))$.

$$\begin{tikzcd} x = \alpha(g) = \beta(g^{-1}) \arrow[r, bend left, "g"] & y = \beta(g) = \alpha(g^{-1}) \arrow[l, bend left, "g^{-1}"] \end{tikzcd}$$

A groupoid G over a set M will be denoted simply by the symbol $G \rightrightarrows M$.

The groupoid $G \rightrightarrows M$ is said to be a ***Lie groupoid*** if G and M are manifolds and all the structural maps are differentiable with α and β differentiable submersions. If $G \rightrightarrows M$ is a Lie groupoid then m is a submersion, ϵ is an immersion and i is a diffeomorphism. Moreover, if $x \in M$, $\alpha^{-1}(x)$ (resp., $\beta^{-1}(x)$) will be said the α***-fiber*** (resp., the β***-fiber***) of x.

On the other hand, if $g \in G$ then the ***left-translation by*** $g \in G$ and the ***right-translation by*** g are the diffeomorphisms

$$\begin{array}{lcl} l_g : \alpha^{-1}(\beta(g)) \longrightarrow \alpha^{-1}(\alpha(g)) & ; & h \longrightarrow l_g(h) = gh, \\ r_g : \beta^{-1}(\alpha(g)) \longrightarrow \beta^{-1}(\beta(g)) & ; & h \longrightarrow r_g(h) = hg. \end{array}$$

Note that $l_g^{-1} = l_{g^{-1}}$ and $r_g^{-1} = r_{g^{-1}}$.

A vector field $\tilde{X}$ on G is said to be ***left-invariant*** (resp., ***right-invariant***) if it is tangent to the fibers of α (resp., β) and $\tilde{X}(gh) = (T_h l_g)(\tilde{X}_h)$ (resp., $\tilde{X}(gh) = (T_g r_h)(\tilde{X}(g))$), for $(g,h) \in G_2$.

Now, we will recall the definition of the ***Lie algebroid associated with*** G.

We consider the vector bundle $\tau : AG \to M$, whose fiber at a point $x \in M$ is $A_xG = V_{\epsilon(x)}\alpha = Ker(T_{\epsilon(x)}\alpha)$. It is easy to prove that there exists a bijection between the space $\Gamma(\tau)$ and the set of left-invariant (resp., right-invariant) vector fields on G. If X is a section of $\tau : AG \to M$, the corresponding left-invariant (resp., right-invariant) vector field on G will be denoted $\overleftarrow{X}$ (resp., $\overrightarrow{X}$), where

$$\overleftarrow{X}(g) = (T_{\epsilon(\beta(g))} l_g)(X(\beta(g))), \tag{2.6}$$

$$\overrightarrow{X}(g) = -(T_{\epsilon(\alpha(g))} r_g)((T_{\epsilon(\alpha(g))} i)(X(\alpha(g)))), \tag{2.7}$$

for $g \in G$. Using the above facts, we may introduce a Lie algebroid structure $([\![\cdot,\cdot]\!], \rho)$ on AG, which is defined by

$$\overleftarrow{[\![X,Y]\!]} = [\overleftarrow{X}, \overleftarrow{Y}], \quad \rho(X)(x) = (T_{\epsilon(x)}\beta)(X(x)), \tag{2.8}$$

for $X, Y \in \Gamma(\tau)$ and $x \in M$. Note that

$$\overrightarrow{[\![X,Y]\!]} = -[\overrightarrow{X}, \overrightarrow{Y}], \quad [\overrightarrow{X}, \overleftarrow{Y}] = 0, \tag{2.9}$$

$$Ti \circ \overrightarrow{X} = -\overleftarrow{X} \circ i, \quad Ti \circ \overleftarrow{X} = -\overrightarrow{X} \circ i, \tag{2.10}$$

(for more details, see [4, 17]).

Given two Lie groupoids $G \rightrightarrows M$ and $G' \rightrightarrows M'$, a ***morphism of Lie groupoids*** is a smooth map $\Phi : G \to G'$ such that

$$(g,h) \in G_2 \Longrightarrow (\Phi(g), \Phi(h)) \in (G')_2$$

and

$$\Phi(gh) = \Phi(g)\Phi(h).$$

A morphism of Lie groupoids $\Phi : G \to G'$ induces a smooth map $\Phi_0 : M \to M'$ in such a way that

$$\alpha' \circ \Phi = \Phi_0 \circ \alpha, \quad \beta' \circ \Phi = \Phi_0 \circ \beta, \quad \Phi \circ \epsilon = \epsilon' \circ \Phi_0,$$

α, β and ϵ (resp., α', β' and ϵ') being the source, the target and the identity section of G (resp., G').

Suppose that (Φ, Φ_0) is a morphism between the Lie groupoids $G \rightrightarrows M$ and $G' \rightrightarrows M'$ and that $\tau : AG \to M$ (resp., $\tau' : AG' \to M'$) is the Lie algebroid of G (resp., G'). Then, if $x \in M$ we may consider the linear map $A_x(\Phi) : A_xG \to A_{\Phi_0(x)}G'$ defined by

$$A_x(\Phi)(v_{\epsilon(x)}) = (T_{\epsilon(x)}\Phi)(v_{\epsilon(x)}), \quad \text{for } v_{\epsilon(x)} \in A_xG. \tag{2.11}$$

In fact, we have that the pair $(A(\Phi), \Phi_0)$ is a morphism between the Lie algebroids $\tau : AG \to M$ and $\tau' : AG' \to M'$ (see [17]).

Next, we will present some examples of Lie groupoids.

1.- **Lie groups**. Any Lie group G is a Lie groupoid over $\{\mathfrak{e}\}$, the identity element of G. The Lie algebroid associated with G is just the Lie algebra $\mathfrak{g}$ of G.

2.- **The pair or banal groupoid**. Let M be a manifold. The product manifold $M \times M$ is a Lie groupoid over M in the following way: α is the projection onto the first factor and β is the projection onto the second factor; $\epsilon(x) = (x,x)$, for all $x \in M$, $m((x,y),(y,z)) = (x,z)$, for $(x,y),(y,z) \in M \times M$ and $i(x,y) = (y,x)$. $M \times M \rightrightarrows M$ is called the ***pair or banal groupoid***. If x is a point of M, it follows that

$$V_{\epsilon(x)}\alpha = \{0_x\} \times T_xM \subseteq T_xM \times T_xM \cong T_{(x,x)}(M \times M).$$

Thus, the linear maps

$$\Psi_x : T_xM \to V_{\epsilon(x)}\alpha, \quad v_x \to (0_x, v_x),$$

induce an isomorphism (over the identity of M) between the Lie algebroids $\tau_M : TM \to M$ and $\tau : A(M \times M) \to M$.

3.- **The Lie groupoid associated with a fibration**. Let $\pi : P \to M$ be a fibration, that is, π is a surjective submersion and denote by G_π the subset of $P \times P$ given by

$$G_\pi = \{(p,p') \in P \times P / \pi(p) = \pi(p')\}.$$

Then, G_π is a Lie groupoid over P and the structural maps α_π, β_π, m_π, ϵ_π and i_π are the restrictions to G_π of the structural maps of the pair groupoid $P \times P \rightrightarrows P$.

If p is a point of P it follows that

$$V_{\epsilon_\pi(p)}\alpha_\pi = \{(0_p, Y_p) \in T_pP \times T_pP / (T_p\pi)(Y_p) = 0\}.$$

Thus, if $(\tau_P)_{|V\pi} : V\pi \to P$ is the vertical bundle to π then the linear maps

$$(\Psi_\pi)_p : V_p\pi \longrightarrow V_{\epsilon_\pi(p)}\alpha_\pi, \quad Y_p \longrightarrow (0_p, Y_p)$$

induce an isomorphism (over the identity of M) between the Lie algebroids $(\tau_P)_{|V\pi} : V\pi \to P$ and $\tau : AG_\pi \to P$.

4.- **Atiyah or gauge groupoids**. Let $p : Q \to M$ be a principal G-bundle. Then, the free action, $\Phi : G \times Q \to Q$, $(g,q) \to \Phi(g,q) = gq$, of G on Q induces, in a natural way, a free action $\Phi \times \Phi : G \times (Q \times Q) \to Q \times Q$ of G on $Q \times Q$ given by $(\Phi \times \Phi)(g,(q,q')) = (gq, gq')$, for $g \in G$ and $(q,q') \in Q \times Q$. Moreover, one may consider the quotient manifold $(Q \times Q)/G$ and it admits a Lie groupoid structure over M with structural maps given by

$$
\begin{array}{lll}
\tilde{\alpha} : (Q \times Q)/G \longrightarrow M & ; & [(q, q')] \longrightarrow p(q), \\
\tilde{\beta} : (Q \times Q)/G \longrightarrow M & ; & [(q, q')] \longrightarrow p(q'), \\
\tilde{\epsilon} : M \longrightarrow (Q \times Q)/G & ; & x \longrightarrow [(q, q)], \quad \text{if } p(q) = x, \\
\tilde{m} : ((Q \times Q)/G)_2 \longrightarrow (Q \times Q)/G & ; & ([(q, q')], [(gq', q'')]) \longrightarrow [(gq, q'')], \\
\tilde{i} : (Q \times Q)/G \longrightarrow (Q \times Q)/G & ; & [(q, q')] \longrightarrow [(q', q)].
\end{array}
$$

This Lie groupoid is called ***the Atiyah (gauge) groupoid associated with the principal G-bundle*** $p : Q \to M$ (see [16]).

If x is a point of M such that $p(q) = x$, with $q \in Q$, and $p_{Q\times Q} : Q \times Q \to (Q \times Q)/G$ is the canonical projection then it is clear that

$$V_{\tilde{\epsilon}(x)}\tilde{\alpha} = (T_{(q,q)}p_{Q\times Q})(\{0_q\} \times T_q Q).$$

Thus, if $\tau_Q|G : TQ/G \to M$ is the Atiyah algebroid associated with the principal G-bundle $p : G \to M$ then the linear maps

$$(TQ/G)_x \to V_{\tilde{\epsilon}(x)}\tilde{\alpha} \;\; ; \;\; [v_q] \to (T_{(q,q)}p_{Q\times Q})(0_q, v_q), \text{ with } v_q \in T_q Q,$$

induce an isomorphism (over the identity of M) between the Lie algebroids $\tau : A((Q \times Q)/G) \to M$ and $\tau_Q|G : TQ/G \to M$.

5.- **The prolongation of a Lie groupoid over a fibration**. Given a Lie groupoid $G \rightrightarrows M$ and a fibration $\pi : P \to M$, we consider the set

$$\mathcal{P}^\pi G = P \;{}_\pi\!\times_\alpha G \;{}_\beta\!\times_\pi P = \{(p, g, p') \in P \times G \times P / \pi(p) = \alpha(g), \;\; \beta(g) = \pi(p')\}.$$

Then, $\mathcal{P}^\pi G$ is a Lie groupoid over P with structural maps given by

$$
\begin{array}{lll}
\alpha^\pi : \mathcal{P}^\pi G \longrightarrow P & ; & (p, g, p') \longrightarrow p, \\
\beta^\pi : \mathcal{P}^\pi G \longrightarrow P & ; & (p, g, p') \longrightarrow p', \\
\epsilon^\pi : P \longrightarrow \mathcal{P}^\pi G & ; & p \longrightarrow (p, \epsilon(\pi(p)), p), \\
m^\pi : (\mathcal{P}^\pi G)_2 \longrightarrow \mathcal{P}^\pi G & ; & ((p, g, p'), (p', h, p'')) \longrightarrow (p, gh, p''), \\
i^\pi : \mathcal{P}^\pi G \longrightarrow \mathcal{P}^\pi G & ; & (p, g, p') \longrightarrow (p', g^{-1}, p).
\end{array}
$$

$\mathcal{P}^\pi G$ is called the ***prolongation of*** G ***over*** $\pi : P \to M$.

Now, denote by $\tau : AG \to M$ the Lie algebroid of G, by $A(\mathcal{P}^\pi G)$ the Lie algebroid of $\mathcal{P}^\pi G$ and by $\mathcal{P}^\pi(AG)$ the prolongation of $\tau : AG \to M$ over the fibration π. If $p \in P$ and $m = \pi(p)$, then it follows that

$$A_p(\mathcal{P}^\pi G) = \{(0_p, v_{\epsilon(m)}, X_p) \in T_p P \times A_m G \times T_p P / (T_p\pi)(X_p) = (T_{\epsilon(m)}\beta)(v_{\epsilon(m)})\}$$

and, thus, one may consider the linear isomorphism

$$(\Psi^\pi)_p : A_p(\mathcal{P}^\pi G) \longrightarrow \mathcal{P}^\pi_p(AG), \quad (0_p, v_{\epsilon(m)}, X_p) \longrightarrow (v_{\epsilon(m)}, X_p). \tag{2.12}$$

In addition, one may prove that the maps $(\Psi^\pi)_p$, $p \in P$, induce an isomorphism $\Psi^\pi : A(\mathcal{P}^\pi G) \to \mathcal{P}^\pi(AG)$ between the Lie algebroids $A(\mathcal{P}^\pi G)$ and $\mathcal{P}^\pi(AG)$ (for more details, see [9]).

6.- **Action Lie groupoids**. Let $G \rightrightarrows M$ be a Lie groupoid and $\pi : P \to M$ be a smooth map. If $P \;{}_\pi\!\times_\alpha G = \{(p, g) \in P \times G / \pi(p) = \alpha(g)\}$ then a right action of G on π is a smooth map

$$P \;{}_\pi\!\times_\alpha G \to P, \;\; (p, g) \to pg,$$

which satisfies the following relations

$$
\begin{array}{rcll}
\pi(pg) & = & \beta(g), & \text{for } (p, g) \in P \;{}_\pi\!\times_\alpha G, \\
(pg)h & = & p(gh), & \text{for } (p, g) \in P \;{}_\pi\!\times_\alpha G \text{ and } (g, h) \in G_2, \text{ and} \\
p\epsilon(\pi(p)) & = & p, & \text{for } p \in P.
\end{array}
$$

Given such an action one constructs ***the action Lie groupoid*** $P\,{}_\pi\!\times_\alpha G$ over P by defining

$$\begin{array}{lcl}
\tilde{\alpha}_\pi : P\,{}_\pi\!\times_\alpha G \longrightarrow P & ; & (p,g) \longrightarrow p, \\
\tilde{\beta}_\pi : P\,{}_\pi\!\times_\alpha G \longrightarrow P & ; & (p,g) \longrightarrow pg, \\
\tilde{\epsilon}_\pi : P \longrightarrow P\,{}_\pi\!\times_\alpha G & ; & p \longrightarrow (p, \epsilon(\pi(p))), \\
\tilde{m}_\pi : (P\,{}_\pi\!\times_\alpha G)_2 \longrightarrow P\,{}_\pi\!\times_\alpha G & ; & ((p,g),(pg,h)) \longrightarrow (p, gh), \\
\tilde{i}_\pi : P\,{}_\pi\!\times_\alpha G \longrightarrow P\,{}_\pi\!\times_\alpha G & ; & (p,g) \longrightarrow (pg, g^{-1}).
\end{array}$$

Now, if $p \in P$, we consider the map $p\,\cdot : \alpha^{-1}(\pi(p)) \to P$ given by

$$p \cdot (g) = pg.$$

Then, if $\tau : AG \to M$ is the Lie algebroid of G, the $\mathbb{R}$-linear map $\Phi : \Gamma(\tau) \to \mathfrak{X}(P)$ defined by

$$\Phi(X)(p) = (T_{\epsilon(\pi(p))} p\, \cdot)(X(\pi(p))), \quad \text{for } X \in \Gamma(\tau) \text{ and } p \in P,$$

induces an action of AG on $\pi : P \to M$. In addition, the Lie algebroid associated with the Lie groupoid $P\,{}_\pi\!\times_\alpha G \rightrightarrows P$ is the action Lie algebroid $AG \ltimes \pi$ (for more details, see [9]).

3. Lie algebroid structure on the vector bundle $\pi^\tau : \mathcal{P}^\tau G \to G$

Let $G \rightrightarrows M$ be a Lie groupoid with structural maps

$$\alpha, \beta : G \to M, \;\; \epsilon : M \to G, \;\; i : G \to G, \;\; m : G_2 \to G.$$

Suppose that $\tau : AG \to M$ is the Lie algebroid of G and that $\mathcal{P}^\tau G$ is the prolongation of G over the fibration $\tau : AG \to M$ (see Example 5 in Section 2.2), that is,

$$\mathcal{P}^\tau G = AG\,{}_\tau\!\times_\alpha G\,{}_\beta\!\times_\tau AG.$$

$\mathcal{P}^\tau G$ is a Lie groupoid over AG and we may define the bijective map $\Theta : \mathcal{P}^\tau G \to V\beta \oplus_G V\alpha$ as follows

$$\Theta(u_{\epsilon(\alpha(g))}, g, v_{\epsilon(\beta(g))}) = ((T_{\epsilon(\alpha(g))}(r_g \circ i))(u_{\epsilon(\alpha(g))}), (T_{\epsilon(\beta(g))} l_g)(v_{\epsilon(\beta(g))})),$$

for $(u_{\epsilon(\alpha(g))}, g, v_{\epsilon(\beta(g))}) \in A_{\alpha(g)}G \times G \times A_{\beta(g)}G$. Thus, $V\beta \oplus_G V\alpha$ is a Lie groupoid over AG (this Lie groupoid was considered by Saunders [30]). We remark that the Lie algebroid of $\mathcal{P}^\tau G \equiv V\beta \oplus_G V\alpha \rightrightarrows AG$ is isomorphic to the prolongation of AG over $\tau : AG \to M$ and that the prolongation of a Lie algebroid A over the vector bundle projection $\tau : A \to M$ plays an important role in the description of Lagrangian Mechanics on A (see [14, 22]).

On the other hand, note that $\mathcal{P}^\tau G \equiv V\beta \oplus_G V\alpha$ is a real vector bundle over G. In this section, we will prove that the vector bundle $\pi^\tau : \mathcal{P}^\tau G \to G$ admits an integrable Lie algebroid structure. In other words, we will prove that there exists a Lie groupoid $H \rightrightarrows G$ over G such that the Lie algebroid AH is isomorphic to the real vector bundle $\pi^\tau : \mathcal{P}^\tau G \to G$. In addition, we will see that the Lie groupoid H is isomorphic to the prolongations of G over α and β.

It is clear that the Lie algebroids of the Lie groupoids over G

$$G_\beta = \{(g,h) \in G \times G / \beta(g) = \beta(h)\}, \;\; G_\alpha = \{(r,s) \in G \times G / \alpha(r) = \alpha(s)\},$$

are just $V\beta \to G$ and $V\alpha \to G$, respectively. This fact suggests to consider the following manifold

$$G_\beta \star G_\alpha = \{((g,h),(r,s)) \in G_\beta \times G_\alpha / \beta_\beta(g,h) = \alpha_\alpha(r,s)\},$$

where $\beta_\beta : G_\beta \to G$ (respectively, $\alpha_\alpha : G_\alpha \to G$) is the target (respectively, the source) of the Lie groupoid $G_\beta \rightrightarrows G$ (respectively, $G_\alpha \rightrightarrows G$).

We will identify the space $G_\beta \star G_\alpha$ with

$$\{(g,h,s) \in G \times G \times G / \beta(g) = \beta(h), \; \alpha(h) = \alpha(s)\}.$$

This space admits a Lie groupoid structure over G with structural maps given by

$$\begin{array}{ll}\alpha_{\beta\alpha} : G_\beta \star G_\alpha \longrightarrow G & ; (g,h,s) \longrightarrow g,\\ \beta_{\beta\alpha} : G_\beta \star G_\alpha \longrightarrow G & ; (g,h,s) \longrightarrow s,\\ \epsilon_{\beta\alpha} : G \longrightarrow G_\beta \star G_\alpha & ; g \longrightarrow (g,g,g),\\ m_{\beta\alpha} : (G_\beta \star G_\alpha)_2 \longrightarrow G_\beta \star G_\alpha & ; ((g,h,s),(s,h',s')) \longrightarrow (g,h's^{-1}h,s'),\\ i_{\beta\alpha} : G_\beta \star G_\alpha \longrightarrow G_\beta \star G_\alpha & ; (g,h,s) \longrightarrow (s,gh^{-1}s,g).\end{array} \tag{3.1}$$

Note that

$$\begin{array}{lll} j_\beta : G_\beta \longrightarrow G_\beta \star G_\alpha & ; & (g,h) \longrightarrow j_\beta(g,h) = (g,h,h),\\ j_\alpha : G_\alpha \longrightarrow G_\beta \star G_\alpha & ; & (h,s) \longrightarrow j_\alpha(h,s) = (h,h,s),\end{array}$$

are Lie groupoid morphisms and that the map

$$m_{\beta\alpha}(j_\beta, j_\alpha) : G_\beta \star G_\alpha \to G_\beta \star G_\alpha;\ \ (g,h,s) \to m_{\beta\alpha}(j_\beta(g,h), j_\alpha(h,s))$$

is just the identity map. This implies that (G_β, G_α) is a ***matched pair of Lie groupoids*** in the sense of Mackenzie [18] (see also [25]).

Denote by $([\![\cdot,\cdot]\!], \rho)$ the Lie algebroid structure on $\tau : AG \to M$.

Theorem 3.1. *Let $A(G_\beta \star G_\alpha) \to G$ be the Lie algebroid of the Lie groupoid $G_\beta \star G_\alpha \rightrightarrows G$. Then:*

(i) *The vector bundles $A(G_\beta \star G_\alpha) \to G$ and $\pi^\tau : \mathcal{P}^\tau G \cong V\beta \oplus_G V\alpha \to G$ are isomorphic. Thus, the vector bundle $\pi^\tau : \mathcal{P}^\tau G \cong V\beta \oplus_G V\alpha \to G$ admits a Lie algebroid structure.*

(ii) *The anchor map $\rho^{\mathcal{P}^\tau G}$ of $\pi^\tau : \mathcal{P}^\tau G \cong V\beta \oplus_G V\alpha \to G$ is given by*

$$\rho^{\mathcal{P}^\tau G}(X_g, Y_g) = X_g + Y_g, \quad \textit{for } (X_g, Y_g) \in V_g\beta \oplus V_g\alpha, \tag{3.2}$$

and the Lie bracket $[\![\cdot,\cdot]\!]^{\mathcal{P}^\tau G}$ on the space $\Gamma(\pi^\tau)$ is characterized by the following relation

$$[\![(\overrightarrow{X}, \overleftarrow{Y}), (\overrightarrow{X'}, \overleftarrow{Y'})]\!]^{\mathcal{P}^\tau G} = (-\overrightarrow{[\![X, X']\!]}, \overleftarrow{[\![Y, Y']\!]}), \tag{3.3}$$

for $X, Y, X', Y' \in \Gamma(\tau)$.

Proof. (i) If $g \in G$ then, from (3.1), we deduce that the vector space $V_{\epsilon_{\beta\alpha}(g)}\alpha_{\beta\alpha}$ may be described as follows

$$\begin{aligned} V_{\epsilon_{\beta\alpha}(g)}\alpha_{\beta\alpha} &= \{(0_g, X_g, Z_g) \in T_gG \times T_gG \times T_gG / X_g \in V_g\beta,\ (T_g\alpha)(X_g) = (T_g\alpha)(Z_g)\}\\ &\cong \{(X_g, Z_g) \in T_gG \times T_gG / X_g \in V_g\beta,\ (T_g\alpha)(X_g) = (T_g\alpha)(Z_g)\}.\end{aligned}$$

Now, we will define the linear map $\Psi_g : V_{\epsilon_{\beta\alpha}(g)}\alpha_{\beta\alpha} \to V_g\beta \oplus V_g\alpha \cong \mathcal{P}^\tau_g G$ by

$$\Psi_g(X_g, Z_g) = (X_g, Z_g - X_g). \tag{3.4}$$

It is clear that Ψ_g is a linear isomorphism and

$$\Psi_g^{-1}(X_g, Y_g) = (X_g, X_g + Y_g), \quad \text{for } (X_g, Y_g) \in V_g\beta \oplus V_g\alpha \cong \mathcal{P}^\tau_g G. \tag{3.5}$$

Therefore, the collection of the maps Ψ_g, $g \in G$, induces a vector bundle isomorphism $\Psi : A(G_\beta \star G_\alpha) \to \mathcal{P}^\tau G \cong V\beta \oplus_G V\alpha$ over the identity of G.

(ii) A direct computation, using (3.1), proves that the linear map $T_{\epsilon_{\beta\alpha}(g)}\beta_{\beta\alpha} : V_{\epsilon_{\beta\alpha}(g)}\alpha_{\beta\alpha} \to T_gG$ is given by

$$(T_{\epsilon_{\beta\alpha}(g)}\beta_{\beta\alpha})(X_g, Z_g) = Z_g. \tag{3.6}$$

Consequently, from (2.8), (3.5) and (3.6), we deduce that (3.2) holds.

Next, we will prove (3.3).

Using (3.5), it follows that

$$(\Psi^{-1} \circ (\overrightarrow{X}, \overleftarrow{Y}))(g) = (0_g, \overrightarrow{X}(g), \overrightarrow{X}(g) + \overleftarrow{Y}(g)) \cong (\overrightarrow{X}(g), \overrightarrow{X}(g) + \overleftarrow{Y}(g)),$$

for $g\in G$. Denote by $\overleftarrow{\Psi^{-1}\circ(\overrightarrow{X},\overleftarrow{Y})}$ the corresponding left-invariant vector field on $G_\beta\star G_\alpha$. Then, from (2.6) and (3.1), we have that

$$\overleftarrow{\Psi^{-1}\circ(\overrightarrow{X},\overleftarrow{Y})}(g,h,s)=(0_g,\overrightarrow{X}(h),\overrightarrow{X}(s)+\overleftarrow{Y}(s)),\quad \text{for } (g,h,s)\in G_\beta\star G_\alpha.$$

Thus, using (2.8) and (2.9), we conclude that

$$[\overleftarrow{\Psi^{-1}\circ(\overrightarrow{X},\overleftarrow{Y})},\overleftarrow{\Psi^{-1}\circ(\overrightarrow{X'},\overleftarrow{Y'})}]=\overleftarrow{\Psi^{-1}\circ(-\overrightarrow{[\![X,X']\!]},\overleftarrow{[\![Y,Y']\!]})}.$$

Therefore, we obtain that (3.3) holds. $\square$

The above diagram shows the Lie groupoid and Lie algebroid structures of $\mathcal{P}^\tau G$:

$$\begin{tikzcd}
\mathcal{P}^\tau G \arrow[rr, shift left, "\alpha^\tau"] \arrow[rr, shift right, "\beta^\tau"'] \arrow[dr, "\rho^{\mathcal{P}^\tau G}"] \arrow[dd, "\pi^\tau"'] & & AG \arrow[dr, "\rho"] \arrow[dd, "\tau"] & \\
& TG \arrow[rr, shift left, "T\alpha"] \arrow[rr, shift right, "T\beta"'] \arrow[dl, "\tau_G"] & & TM \arrow[dl, "\tau_M"] \\
G \arrow[rr, shift left, "\alpha"] \arrow[rr, shift right, "\beta"'] & & M &
\end{tikzcd}$$

Given a section X of $AG\longrightarrow M$, we define the sections $X^{(1,0)}$, $X^{(0,1)}$ (the β and α- lifts) and $X^{(1,1)}$ (the complete lift) of X to $\pi^\tau:\mathcal{P}^\tau G\longrightarrow G$ as follows:

$$X^{(1,0)}(g)=(\overrightarrow{X}(g),0_g),\quad X^{(0,1)}(g)=(0_g,\overleftarrow{X}(g))\quad\text{and}\quad X^{(1,1)}(g)=(-\overrightarrow{X}(g),\overleftarrow{X}(g))$$

We can easily see that

$$\begin{array}{l} [\![X^{(1,0)},Y^{(1,0)}]\!]^{\mathcal{P}^\tau G}=-[\![X,Y]\!]^{(1,0)}\\ [\![X^{(0,1)},Y^{(0,1)}]\!]^{\mathcal{P}^\tau G}=[\![X,Y]\!]^{(0,1)}\end{array}\quad\text{and}\quad [\![X^{(0,1)},Y^{(1,0)}]\!]^{\mathcal{P}^\tau G}=0 \tag{3.7}$$

and, as a consequence,

$$\begin{array}{l} [\![X^{(1,1)},Y^{(1,0)}]\!]^{\mathcal{P}^\tau G}=[\![X,Y]\!]^{(1,0)}\\ [\![X^{(1,1)},Y^{(0,1)}]\!]^{\mathcal{P}^\tau G}=[\![X,Y]\!]^{(0,1)}\end{array}\quad\text{and}\quad [\![X^{(1,1)},Y^{(1,1)}]\!]^{\mathcal{P}^\tau G}=[\![X,Y]\!]^{(1,1)}. \tag{3.8}$$

Remark 3.2. From Theorem 3.1, we deduce that the canonical inclusions

$$(Id,0):V\beta\to\mathcal{P}^\tau G\cong V\beta\oplus_G V\alpha,\quad (0,Id):V\alpha\to\mathcal{P}^\tau G\cong V\beta\oplus_G V\alpha,$$

are Lie algebroid morphisms over the identity of G. In other words, $(V\beta,V\alpha)$ is a ***matched pair of Lie algebroids*** in the sense of Mokri [25]. This fact directly follows using the following general theorem (see [25]): if (G,H) is a matched pair of Lie groupoids then (AG,AH) is a matched pair of Lie algebroids. $\diamond$

Next, we will consider the prolongation $\mathcal{P}^\beta G$ of the Lie groupoid G over the target $\beta:G\to M$. We recall that

$$\mathcal{P}^\beta G=G\;{}_\beta{\times}_\alpha\;G\;{}_\beta{\times}_\beta\;G=\{(g,h,s)\in G\times G\times G/\beta(g)=\alpha(h),\ \beta(h)=\beta(s)\},$$

and that $\mathcal{P}^\beta G$ is a Lie groupoid over G with structural maps

$$\begin{array}{lcl}\alpha^\beta:\mathcal{P}^\beta G\longrightarrow G & ; & (g,h,s)\longrightarrow g,\\ \beta^\beta:\mathcal{P}^\beta G\longrightarrow G & ; & (g,h,s)\longrightarrow s,\\ \epsilon^\beta:G\longrightarrow\mathcal{P}^\beta G & ; & g\longrightarrow(g,\epsilon(\beta(g)),g),\\ m^\beta:(\mathcal{P}^\beta G)_2\longrightarrow\mathcal{P}^\beta G & ; & ((g,h,s),(s,t,u))\longrightarrow(g,ht,u),\\ i^\beta:\mathcal{P}^\beta G\longrightarrow\mathcal{P}^\beta G & ; & (g,h,s)\longrightarrow(s,h^{-1},g).\end{array} \tag{3.9}$$

Moreover, we also have that the Lie algebroid of $\mathcal{P}^\beta G$ may be identified with the prolongation $\mathcal{P}^\beta(AG)$ of AG over $\beta:G\to M$. We remark that

$$\mathcal{P}^\beta_g(AG)=\{(v_{\epsilon(\beta(g))},X_g)\in A_{\beta(g)}G\times T_gG/(T_{\epsilon(\beta(g))}\beta)(v_{\epsilon(\beta(g))})=(T_g\beta)(X_g)\}$$

for $g\in G$.

Theorem 3.3. *Let $\Phi^\beta : G_\beta \star G_\alpha \to \mathcal{P}^\beta G$ be the map defined by*

$$\Phi^\beta(g,h,s) = (g, h^{-1}s, s), \tag{3.10}$$

for $(g,h,s) \in G_\beta \star G_\alpha$. Then:

(i) *Φ^β is a Lie groupoid isomorphism over the identity of G.*

(ii) *If $A(\Phi^\beta) : A(G_\beta \star G_\alpha) \to A(\mathcal{P}^\beta G)$ is the corresponding Lie algebroid isomorphism then, under the identifications*

$$A(G_\beta \star G_\alpha) \cong \mathcal{P}^\tau G \cong V\beta \oplus_G V\alpha, \quad A(\mathcal{P}^\beta G) \cong \mathcal{P}^\beta(AG),$$

$A(\Phi^\beta)$ is given by

$$A_g(\Phi^\beta)(X_g, Y_g) = ((T_g l_{g^{-1}})(Y_g), X_g + Y_g), \tag{3.11}$$

for $(X_g, Y_g) \in V_g\beta \oplus V_g\alpha$, where $l_{g^{-1}} : \alpha^{-1}(\alpha(g)) \to \alpha^{-1}(\beta(g))$ is the left-translation by g^{-1}.

Proof. (i) A direct computation, using (3.1) and (3.9), proves the result.

(ii) If $g \in G$ we have that

$$\begin{array}{lcl} A_g(G_\beta \star G_\alpha) = V_{\epsilon_{\beta\alpha}(g)}\alpha_{\beta\alpha} & = & \{(0_g, X_g, Z_g) \in T_gG \times T_gG \times T_gG / X_g \in V_g\beta, \\ & & (T_g\alpha)(X_g) = (T_g\alpha)(Z_g)\}, \\ A_g(\mathcal{P}^\beta G) = V_{\epsilon^\beta(g)}\alpha^\beta & = & \{(0_g, v_{\epsilon(\beta(g))}, Y_g) \in T_gG \times A_{\beta(g)}G \times T_gG / \\ & & (T_g\beta)(Y_g) = (T_{\epsilon(\beta(g))}\beta)(v_{\epsilon(\beta(g))})\}. \end{array}$$

Now, if $(0_g, X_g, Z_g) \in V_{\epsilon_{\beta\alpha}(g)}\alpha_{\beta\alpha}$ then, from (3.10), we deduce that

$$\begin{aligned} (T_{\epsilon_{\beta\alpha}(g)}\Phi^\beta)(0_g, X_g, Z_g) &= (T_{\epsilon_{\beta\alpha}(g)}\Phi^\beta)(0_g, 0_g, Z_g - X_g) + (T_{\epsilon_{\beta\alpha}(g)}\Phi^\beta)(0_g, X_g, X_g) \\ &= (0_g, (T_g l_{g^{-1}})(Z_g - X_g), Z_g - X_g) + (T_{\epsilon_{\beta\alpha}(g)}\Phi^\beta)(0_g, X_g, X_g). \end{aligned}$$

On the other hand, suppose that $\beta(g) = x \in M$ and that $\gamma : (-\varepsilon, \varepsilon) \to \beta^{-1}(x)$ is a curve in $\beta^{-1}(x)$ such that $\gamma(0) = g$ and $\gamma'(0) = X_g$. Then, one may consider the curve $\tilde{\gamma} : (-\varepsilon, \varepsilon) \to G_\beta \star G_\alpha$ on $G_\beta \star G_\alpha$ given by

$$\tilde{\gamma}(t) = (g, \gamma(t), \gamma(t))$$

and it follows that

$$\tilde{\gamma}(0) = (g, g, g), \quad \tilde{\gamma}'(0) = (0_g, X_g, X_g).$$

Moreover, we obtain that

$$\bar{\gamma}(t) = (\Phi^\beta \circ \tilde{\gamma})(t) = (g, \epsilon(x), \gamma(t)), \quad \text{for all } t$$

and thus

$$\bar{\gamma}'(0) = (0_g, 0_{\epsilon(\beta(g))}, X_g).$$

This proves that

$$(T_{\epsilon_{\beta\alpha}(g)}\Phi^\beta)(0_g, X_g, Z_g) = (0_g, (T_g l_{g^{-1}})(Z_g - X_g), Z_g). \tag{3.12}$$

Finally, using (2.11), (2.12), (3.5) and (3.12), we deduce that (3.11) holds. □

Next, we will consider the prolongation $\mathcal{P}^\alpha G$ of the Lie groupoid G over the source $\alpha : G \to M$. We recall that

$$\mathcal{P}^\alpha G = G\ {}_\alpha\!\times_\alpha G\ {}_\beta\!\times_\alpha G = \{(g,h,s) \in G \times G \times G / \alpha(g) = \alpha(h), \beta(h) = \alpha(s)\}$$

and that $\mathcal{P}^\alpha G$ is a Lie groupoid over G with structural maps

$$\begin{array}{lcl} \alpha^\alpha : \mathcal{P}^\alpha G \longrightarrow G & ; & (g,h,s) \longrightarrow g, \\ \beta^\alpha : \mathcal{P}^\alpha G \longrightarrow G & ; & (g,h,s) \longrightarrow s, \\ \epsilon^\alpha : G \longrightarrow \mathcal{P}^\alpha G & ; & g \longrightarrow (g, \epsilon(\alpha(g)), g), \\ m^\alpha : (\mathcal{P}^\alpha G)_2 \longrightarrow \mathcal{P}^\alpha G & ; & ((g,h,s),(s,t,u)) \longrightarrow (g, ht, u), \\ i^\alpha : \mathcal{P}^\alpha G \longrightarrow \mathcal{P}^\alpha G & ; & (g,h,s) \longrightarrow (s, h^{-1}, g). \end{array}$$

Moreover, we also have that the Lie algebroid of $\mathcal{P}^\alpha G$ may be identified with the prolongation $\mathcal{P}^\alpha(AG)$ of AG over $\alpha : G \to M$. We remark that

$$\mathcal{P}^\alpha_g(AG) = \{(v_{\epsilon(\alpha(g))}, X_g) \in A_{\alpha(g)}G \times T_gG / (T_{\epsilon(\alpha(g))}\beta)(v_{\epsilon(\alpha(g))}) = (T_g\alpha)(X_g)\},$$

for $g \in G$.

Theorem 3.4. *Let $\Phi^\alpha : G_\beta \star G_\alpha \to \mathcal{P}^\alpha G$ be the map defined by*

$$\Phi^\alpha(g, h, s) = (g, gh^{-1}, s),$$

for $(g, h, s) \in G_\beta \star G_\alpha$. Then:

(i) *Φ^α is a Lie groupoid isomorphism over the identity of G.*

(ii) *If $A(\Phi^\alpha) : A(G_\beta \star G_\alpha) \to A(\mathcal{P}^\alpha G)$ is the corresponding Lie algebroid isomorphism then, under the canonical identifications*

$$A(G_\beta \star G_\alpha) \cong \mathcal{P}^\tau G \cong V\beta \oplus_G V\alpha, \quad A(\mathcal{P}^\alpha G) \cong \mathcal{P}^\alpha(AG),$$

$A(\Phi^\alpha)$ is given by

$$A_g(\Phi^\alpha)(X_g, Y_g) = (T_g(i \circ r_{g^{-1}})(X_g), X_g + Y_g), \tag{3.13}$$

for $(X_g, Y_g) \in V_g\beta \oplus V_g\alpha$, where $r_{g^{-1}} : \beta^{-1}(\beta(g)) \to \beta^{-1}(\alpha(g))$ is the right-translation by g^{-1}.

Proof. Proceeding as in the proof of Theorem 3.3, we deduce the result. □

4. Mechanics on Lie Groupoids

In this section, we introduce Lagrangian (Hamiltonian) Mechanics on an arbitrary Lie groupoid and we will also analyze its geometrical properties. This construction may be considered as a discrete version of the construction of the Lagrangian (Hamiltonian) Mechanics on Lie algebroids proposed in [22] (see also [14, 23]). We first discuss discrete Euler-Lagrange equations following a similar approach to [31], using a variational procedure. Secondly, we intrinsically define and discuss the discrete Poincaré-Cartan sections, Legendre transformations, regularity of the Lagrangian and Noether's theorem.

4.1. Discrete Euler-Lagrange equations.

Let G be a Lie groupoid with structural maps

$$\alpha, \beta : G \to M, \;\; \epsilon : M \to G, \;\; i : G \to G, \;\; m : G_2 \to G.$$

Denote by $\tau : AG \to M$ the Lie algebroid of G.

A ***discrete Lagrangian*** is a function $L : G \longrightarrow \mathbb{R}$. Fixed $g \in G$, we define the set of admissible sequences with values in G:

$$\mathcal{C}^N_g = \{(g_1, \dots, g_N) \in G^N \;/\; (g_k, g_{k+1}) \in G_2 \text{ for } k = 1, \dots, N-1 \text{ and } g_1 \dots g_n = g\}.$$

Given a tangent vector at $(g_1, \dots, g_N)$ to the manifold $\mathcal{C}^N_g$, we may write it as the tangent vector at $t = 0$ of a curve in $\mathcal{C}^N_g$, $t \in (-\varepsilon, \varepsilon) \subseteq \mathbb{R} \longrightarrow c(t)$ which passes through $(g_1, \dots, g_N)$ at $t = 0$. This type of curves is of the form

$$c(t) = (g_1h_1(t), h_1^{-1}(t)g_2h_2(t), \dots, h_{N-2}^{-1}(t)g_{N-1}h_{N-1}(t), h_{N-1}^{-1}(t)g_N)$$

where $h_k(t) \in \alpha^{-1}(\beta(g_k))$, for all t, and $h_k(0) = \epsilon(\beta(g_k))$ for $k = 1, \dots, N-1$.

Therefore, we may identify the tangent space to $\mathcal{C}^N_g$ at $(g_1, \dots, g_N)$ with

$$T_{(g_1,\dots,g_N)}\mathcal{C}^N_g \equiv \{(v_1, \dots, v_{N-1}) \;/\; v_k \in A_{x_k}G \text{ and } x_k = \beta(g_k), 1 \le k \le N-1\}.$$

Observe that each v_k is the tangent vector to the α-vertical curve h_k at $t = 0$.

The curve c is called a ***variation*** of $(g_1, \dots, g_N)$ and $(v_1, v_2, \dots, v_{N-1})$ is called an ***infinitesimal variation*** of $(g_1, \dots, g_N)$.

Define the ***discrete action sum*** associated to the discrete Lagrangian $L : G \longrightarrow \mathbb{R}$

$$\begin{array}{rcll} \mathcal{S}L & : & \mathcal{C}_g^N & \longrightarrow & \mathbb{R} \\ & & (g_1, \dots, g_N) & \longmapsto & \displaystyle\sum_{k=1}^N L(g_k). \end{array}$$

We now proceed, as in the continuous case, to derive the discrete equations of motion applying Hamilton's principle of critical action. For it, we consider variations of the discrete action sum.

Definition 4.1 (Discrete Hamilton's principle [31])**.** *Given $g \in G$, an admissible sequence $(g_1, \dots, g_N) \in \mathcal{C}_g^N$ is a solution of the Lagrangian system determined by $L : G \longrightarrow \mathbb{R}$ if and only if $(g_1, \dots, g_N)$ is a critical point of $\mathcal{S}L$.*

Fist of all, in order to characterize the critical points, we need to calculate:

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \mathcal{S}L(c(t)) &= \frac{d}{dt}\Big|_{t=0} \big\{ L(g_1 h_1(t)) + L(h_1^{-1}(t) g_2 h_2(t)) \\ &\quad + \dots + L(h_{N-2}^{-1}(t) g_{N-1} h_{N-1}(t)) + L(h_{N-1}^{-1}(t) g_N) \big\} . \end{aligned}$$

Therefore,

$$\frac{d}{dt}\Big|_{t=0} \mathcal{S}L(c(t)) = \sum_{k=1}^{N-1} \big(\mathrm{d}^\circ (L \circ l_{g_k})(\epsilon(x_k))(v_k) + \mathrm{d}^\circ (L \circ r_{g_{k+1}} \circ i)(\epsilon(x_k))(v_k) \big)$$

where d° is the standard differential on G, i.e., d° is the differential of the Lie algebroid $\tau_G : TG \to G$. Since the critical condition is $\frac{d}{dt}\Big|_{t=0} \mathcal{S}L(c(t)) = 0$ then, applying (2.6) and (2.7), we may rewrite this condition as

$$0 = \sum_{k=1}^{N-1} \Big[\overleftarrow{X}_k (g_k)(L) - \overrightarrow{X}_k (g_{k+1})(L) \Big] = \sum_{k=1}^{N-1} \Big[\langle dL, X_k^{(0,1)} \rangle (g_k) - \langle dL, X_k^{(1,0)} \rangle (g_{k+1}) \Big]$$

where d is the differential of the Lie algebroid $\pi^\tau : \mathcal{P}^\tau G \equiv V\beta \oplus_G V\alpha \longrightarrow G$ and X_k is a section of $\tau : AG \to M$ such that $X_k(x_k) = v_k$.

For $N = 2$ we obtain that $(g_1, g_2) \in G_2$ (with $\beta(g_1) = \alpha(g_2) = x$) is a solution if

$$\mathrm{d}^\circ \left[L \circ l_{g_1} + L \circ r_{g_2} \circ i \right] (\epsilon(x))_{|A_x G} = 0$$

or, alternatively,

$$\overleftarrow{X}(g_1)(L) - \overrightarrow{X}(g_2)(L) = 0$$

for every section X of AG. These equations will be called ***discrete Euler-Lagrange equations***.

Thus, we may define the ***discrete Euler-Lagrange operator***:

$$D_{\mathrm{DEL}} L : G_2 \longrightarrow A^* G \ ,$$

where A^*G is the dual of AG. This operator is given by

$$D_{\mathrm{DEL}} L(g, h) = d^0 \left[L \circ l_g + L \circ r_h \circ i \right] (\epsilon(x))_{|A_x G}$$

with $\beta(g) = \alpha(h) = x$.

In conclusion, we have characterized the solutions of the Lagrangian system determined by $L : G \longrightarrow \mathbb{R}$ as the sequences $(g_1, \dots, g_N)$, with $(g_k, g_{k+1}) \in G_2$, for each $k \in \{1, \dots, N-1\}$, and

$$D_{\mathrm{DEL}} L(g_k, g_{k+1}) = 0, \quad 1 \leq k \leq N-1.$$

4.2. **Discrete Poincaré-Cartan sections.** Given a Lagrangian function $L : G \longrightarrow \mathbb{R}$, we will study the geometrical properties of the discrete Euler-Lagrange equations.

Consider the Lie algebroid $\pi^\tau : P^\tau G \cong V\beta \oplus_G V\alpha \longrightarrow G$, and define the ***Poincaré-Cartan 1-sections*** $\Theta_L^-, \Theta_L^+ \in \Gamma((\pi^\tau)^*)$ as follows

$$\Theta_L^-(g)(X_g, Y_g) = -X_g(L), \qquad \Theta_L^+(g)(X_g, Y_g) = Y_g(L), \tag{4.1}$$

for each $g \in G$ and $(X_g, Y_g) \in V_g\beta \oplus V_g\alpha$. From the definition, we have that

$$\Theta_L^-(g)(X^{(1,0)}(g)) = -\overrightarrow{X}(g)(L) \quad \text{and} \quad \Theta_L^-(g)(X^{(0,1)}(g)) = 0,$$

and similarly

$$\Theta_L^+(g)(X^{(0,1)}(g)) = \overleftarrow{X}(g)(L) \quad \text{and} \quad \Theta_L^+(g)(X^{(1,0)}(g)) = 0,$$

for $X \in \Gamma(\tau)$.

We also have that $dL = \Theta_L^+ - \Theta_L^-$ and so, using $d^2 = 0$, it follows that $d\Theta_L^+ = d\Theta_L^-$. This means that there exists a unique 2-section $\Omega_L = -d\Theta_L^+ = -d\Theta_L^-$, that will be called the ***Poincaré-Cartan*** 2-section. This 2-section will be important for studying symplecticity of the discrete Euler-Lagrange equations.

Proposition 4.2. *If X and Y are sections of the Lie algebroid AG then*

$$\Omega_L(X^{(1,0)}, Y^{(1,0)}) = 0\ , \ \Omega_L(X^{(0,1)}, Y^{(0,1)}) = 0,$$

and

$$\Omega_L(X^{(1,0)}, Y^{(0,1)}) = -\overrightarrow{X}(\overleftarrow{Y}L) \ \ \textit{and} \ \ \Omega_L(X^{(0,1)}, Y^{(1,0)}) = \overrightarrow{Y}(\overleftarrow{X}L).$$

Proof. A direct computation proves the result. □

Remark 4.3. Remark 4.3. Let g be an element of G such that $\alpha(g) = x$ and $\beta(g) = y$. Suppose that U and V are open subsets of M, with $x \in U$ and $y \in V$, and that $\{X_i\}$ and $\{Y_j\}$ are local bases of $\Gamma(\tau)$ on U and V, respectively. Then, $\{X_i^{(1,0)}, Y_j^{(0,1)}\}$ is a local basis of $\Gamma(\pi^\tau)$ on the open subset $\alpha^{-1}(U) \cap \beta^{-1}(V)$. Moreover, if we denote by $\{(X^i)^{(1,0)}, (Y^j)^{(0,1)}\}$ the dual basis of $\{X_i^{(1,0)}, Y_j^{(0,1)}\}$, we have that on the open subset $\alpha^{-1}(U) \cap \beta^{-1}(V)$

$$\begin{array}{l} \Theta_L^- = -\overrightarrow{X_i}(L)(X^i)^{(1,0)}, \qquad \Theta_L^+ = \overleftarrow{Y_j}(L)(Y^j)^{(0,1)}, \\ \Omega_L = -\overrightarrow{X_i}(\overleftarrow{Y_j}(L))(X^i)^{(1,0)} \wedge (Y^j)^{(0,1)}. \end{array}$$

◇

Finally, we obtain some useful expressions of the Poincaré-Cartan 1-sections using the Lie algebroid isomorphisms introduced in Theorems 3.3 and 3.4.

We recall that the maps

$$\begin{array}{rcl} A(\Phi^\beta) : A(G_\beta * G_\alpha) \cong V\beta \oplus_G V\alpha & \longrightarrow & A(\mathcal{P}^\beta G) \cong \mathcal{P}^\beta(AG) \\ A(\Phi^\alpha) : A(G_\beta * G_\alpha) \cong V\beta \oplus_G V\alpha & \longrightarrow & A(\mathcal{P}^\alpha G) \cong \mathcal{P}^\alpha(AG) \end{array}$$

given by (3.11) and (3.13) are Lie algebroid isomorphisms over the identity of G. Moreover, if $(v_{\epsilon(\beta(g))}, Z_g) \in \mathcal{P}_g^\beta(AG)$ then, from (3.11), it follows that

$$A_g(\Phi^\beta)^{-1}(v_{\epsilon(\beta(g))}, Z_g) = (Z_g - (T_{\epsilon(\beta(g))}l_g)(v_{\epsilon(\beta(g))}), (T_{\epsilon(\beta(g))}l_g)(v_{\epsilon(\beta(g))})). \tag{4.2}$$

On the other hand, if $(v_{\epsilon(\alpha(h))}, Z_h) \in \mathcal{P}_h^\alpha(AG)$ then, using (3.13), we deduce that

$$A_h(\Phi^\alpha)^{-1}(v_{\epsilon(\alpha(h))}, Z_h) = (T_{\epsilon(\alpha(h))}(r_h \circ i)(v_{\epsilon(\alpha(h))}), Z_h - T_{\epsilon(\alpha(h))}(r_h \circ i)(v_{\epsilon(\alpha(h))})). \tag{4.3}$$

Now, we introduce the sections $\Theta_L^\alpha \in \Gamma((\tau^\alpha)^*)$ and $\Theta_L^\beta \in \Gamma((\tau^\beta)^*)$ given by

$$\Theta_L^\alpha = (A(\Phi^\alpha)^{-1}, Id)^*(\Theta_L^-), \quad \Theta_L^\beta = (A(\Phi^\beta)^{-1}, Id)^*(\Theta_L^+). \tag{4.4}$$

Using (4.2) and (4.3), we obtain that

$$\Theta_L^\alpha(h)(v_{\epsilon(\alpha(h))}, Z_h) = -v_{\epsilon(\alpha(h))}(L \circ r_h \circ i), \tag{4.5}$$

$$\Theta_L^\beta(g)(v_{\epsilon(\beta(g))}, Z_g) = v_{\epsilon(\beta(g))}(L \circ l_g), \tag{4.6}$$

for $(v_{\epsilon(\alpha(h))}, Z_h) \in \mathcal{P}_h^\alpha(AG)$ and $(v_{\epsilon(\beta(g))}, Z_g) \in \mathcal{P}_g^\beta(AG)$.

4.2.1. *Poincaré-Cartan* 1*-sections: variational motivation.* Now, we follow a variational procedure to construct the 1-sections Θ_L^+ and Θ_L^-. We begin by calculating the extremals of $\mathcal{S}L$ for variations that do not fix the point $g \in G$. For it, we consider the manifold

$$\mathcal{C}^N = \{(g_1, \dots, g_N) \in G^N \,/ (g_k, g_{k+1}) \in G_2 \text{ for each } k, \;\; 1 \leq k \leq N-1\}.$$

If $c : (-\varepsilon, \varepsilon) \to \mathcal{C}^N$ is a curve in $\mathcal{C}^N$ and $c(0) = (g_1, \dots, g_N)$ then there exist $N+1$ curves $h_k : (-\varepsilon, \varepsilon) \to \alpha^{-1}(\beta(g_k))$, for $0 \leq k \leq N$, with $h_k(0) = \epsilon(\beta(g_k))$ and $g_0 = g_1^{-1}$, such that

$$c(t) = (h_0^{-1}(t) g_1 h_1(t), h_1^{-1}(t) g_2 h_2(t), \dots, h_{N-2}^{-1}(t) g_{N-1} h_{N-1}(t), h_{N-1}^{-1}(t) g_N h_N(t))$$

for $t \in (-\varepsilon, \varepsilon)$. Thus, the tangent space to $\mathcal{C}^N$ at $(g_1, \dots, g_N)$ may be identified with the vector space $A_{\beta(g_0)}G \times A_{\beta(g_1)}G \times \cdots \times A_{\beta(g_N)}G$, that is,

$$T_{(g_1, g_2, \dots, g_N)}\mathcal{C}^N \equiv \{(v_0, v_1, \dots, v_N) \,/\, v_k \in A_{x_k}G, x_k = \beta(g_k), 0 \leq k \leq N\}.$$

Now, proceeding as in Section 4.1, we introduce the action sum

$$\mathcal{S}L : \mathcal{C}^N \longrightarrow \mathbb{R}, \quad \mathcal{S}L(g_1, \dots, g_N) = \sum_{k=1}^N L(g_k).$$

Then,

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \mathcal{S}L(c(t)) &= \sum_{k=1}^{N-1} \left[\mathrm{d}^\circ(L \circ l_{g_k})(\epsilon(x_k))(v_k) + \mathrm{d}^\circ(L \circ r_{g_{k+1}} \circ i)(\epsilon(x_{k+1}))(v_k) \right] \\ &\quad + \mathrm{d}^\circ(L \circ r_{g_1} \circ i)(\epsilon(x_0))(v_0) + \mathrm{d}^\circ(L \circ l_{g_N})(\epsilon(x_N))(v_N). \end{aligned} \tag{4.7}$$

Therefore, if $X_0, \dots, X_N$ are sections of $\tau : AG \to M$ satisfying, $X_k(x_k) = v_k$, for all k, we have that

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \mathcal{S}L(c(t)) &= \sum_{k=1}^{N-1} \left[\overleftarrow{X}_k(g_k)(L) - \overrightarrow{X}_k(g_{k+1})(L) \right] - \overrightarrow{X_0}(g_1)(L) + \overleftarrow{X}_N(g_N)(L) \\ &= \sum_{k=1}^{N-1} (D_{\mathrm{DEL}} L(g_k, g_{k+1}))(v_k) + \Theta_L^-(g_1)(X_0^{(1,0)}(g_1)) + \Theta_L^+(g_N)(X_N^{(0,1)}(g_N)). \end{aligned}$$

Note that it is in the last two terms (that arise from the boundary variations) where appear the Poincaré-Cartan 1-sections.

4.3. **Discrete Lagrangian evolution operator.** We say that a differentiable mapping $\xi : G \longrightarrow G$ is a ***discrete flow*** or a ***discrete Lagrangian evolution operator for*** L if it verifies the following properties:

- graph(ξ) $\subseteq G_2$, that is, $(g, \xi(g)) \in G_2$, $\forall g \in G$ (ξ is a second order operator).
- $(g, \xi(g))$ is a solution of the discrete Euler-Lagrange equations, for all $g \in G$, that is, $(D_{\mathrm{DEL}} L)(g, \xi(g)) = 0$, for all $g \in G$.

In such a case

$$\mathrm{d}^\circ(L\circ l_g + L\circ r_{\xi(g)}\circ i)(\epsilon(\beta(g)))_{|A_{\beta(g)}G} = 0, \quad \text{for all } g\in G \tag{4.8}$$

or, in other terms,

$$\overleftarrow{X}(g)(L) - \overrightarrow{X}(\xi(g))(L) = 0 \tag{4.9}$$

for every section X of AG and every $g\in G$.

Now, we define the prolongation $\mathcal{P}^\tau\xi : V\beta\oplus_G V\alpha \longrightarrow V\beta\oplus_G V\alpha$ of the second order operator $\xi : G\longrightarrow G$ as follows:

$$\mathcal{P}^\tau\xi = A(\Phi^\alpha)^{-1}\circ(Id, T\xi)\circ A(\Phi^\beta) \tag{4.10}$$

with $A(\Phi^\alpha)$ and $A(\Phi^\beta)$ the isomorphisms defined in Theorems 3.3 and 3.4 and $(Id, T\xi) : \mathcal{P}^\beta(AG)\to\mathcal{P}^\alpha(AG)$ the map given by

$$(Id, T\xi)(v_{\epsilon(\beta(g))}, X_g) = (v_{\epsilon(\beta(g))}, (T_g\xi)(X_g)), \text{ for } (v_{\epsilon(\beta(g))}, X_g)\in\mathcal{P}^\beta_g(AG).$$

Since the pair $((Id, T\xi),\xi)$ is a Lie algebroid morphism between the Lie algebroids $\mathcal{P}^\beta(AG)\longrightarrow G$ and $\mathcal{P}^\alpha(AG)\longrightarrow G$ then the pair $(\mathcal{P}^\tau\xi,\xi)$ is also a Lie algebroid morphism

$$\begin{tikzcd}
V\beta\oplus_G V\alpha \arrow[rr, "\mathcal{P}^\tau\xi"] \arrow[d] & & V\beta\oplus_G V\alpha \arrow[d] \\
G \arrow[rr, "\xi"] \arrow[dr, "\beta"'] & & G \arrow[dl, "\alpha"] \\
& M &
\end{tikzcd}$$

From the definition of $\mathcal{P}^\tau\xi$, we deduce that

$$\mathcal{P}^\tau_g\xi(X_g, Y_g) = ((T_g(r_{g\xi(g)}\circ i))(Y_g), (T_g\xi)(X_g) + (T_g\xi)(Y_g) - T_g(r_{g\xi(g)}\circ i)(Y_g)) \tag{4.11}$$

for all $(X_g, Y_g)\in V_g\beta\oplus V_g\alpha$. Moreover, from (2.10) and (4.11), we obtain that

$$\mathcal{P}^\tau\xi(\overrightarrow{X}(g), \overleftarrow{Y}(g)) = (-\overrightarrow{Y}(\xi(g)), (T_g\xi)(\overrightarrow{X}(g) + \overleftarrow{Y}(g)) + \overrightarrow{Y}(\xi(g))) \tag{4.12}$$

for all X, Y sections of AG.

4.4. **Preservation of Poincaré-Cartan sections.** The following result explains the sense in which the discrete Lagrange evolution operator preserves the Poincaré-Cartan 2-section.

Theorem 4.4. *Let $L : G\longrightarrow\mathbb{R}$ be a discrete Lagrangian on a Lie groupoid G. Then:*

(i) *The map ξ is a discrete Lagrangian evolution operator for L if and only if $(\mathcal{P}^\tau\xi,\xi)^*\Theta^-_L = \Theta^+_L$.*
(ii) *The map ξ is a discrete Lagrangian evolution operator for L if and only if $(\mathcal{P}^\tau\xi,\xi)^*\Theta^-_L - \Theta^-_L = dL$.*
(iii) *If ξ is discrete Lagrangian evolution operator then $(\mathcal{P}^\tau\xi,\xi)^*\Omega_L = \Omega_L$.*

Proof. From (4.4), it follows

$$(A(\Phi^\alpha), Id)^*(\Theta^\alpha_L) = \Theta^-_L, \quad (A(\Phi^\beta), Id)^*(\Theta^\beta_L) = \Theta^+_L. \tag{4.13}$$

On the other hand, if $(v_{\epsilon(\beta(g))}, X_g)\in\mathcal{P}^\beta_g(AG)$ then, using (4.5) and (4.6), we have that

$$\{((\mathrm{Id}, T\xi),\xi)^*(\Theta^\alpha_L)\}(g)(v_{\epsilon(\beta(g))}, X_g) = -v_{\epsilon(\beta(g))}(L\circ r_{\xi(g)}\circ i)$$

and

$$\Theta^\beta_L(g)(v_{\epsilon(\beta(g))}, X_g) = v_{\epsilon(\beta(g))}(L\circ l_g).$$

Thus, $((\mathrm{Id}, T\xi), \xi)^*\Theta_L^\alpha = \Theta_L^\beta$ if and only if ξ is a discrete Lagrangian evolution operator for L. Therefore, using this fact and (4.13), we prove (i).

The second property follows from (i) by taking into account that $dL = \Theta_L^+ - \Theta_L^-$. Finally, (iii) follows using (ii) and the fact that $(\mathcal{P}^\tau\xi, \xi)$ is a Lie algebroid morphism. □

Remark 4.5. Now, we present a proof of the preservation of the Poincaré-Cartan 2-section using variational arguments. Given a discrete Lagrangian evolution operator $\xi : G \longrightarrow G$ for L, we may consider the function $\mathcal{S}_\xi L : G \to \mathbb{R}$ given by

$$(\mathcal{S}_\xi L)(g) = L(g) + L(\xi(g)), \quad \text{for } g \in G.$$

If d is the differential on the Lie algebroid $V\beta \oplus_G V\alpha \to G$ and X, Y are sections of $\tau : AG \to M$ then, using (4.1), (4.9) and (4.12), we obtain that

$$\begin{aligned} d(\mathcal{S}_\xi L)(g)(\overrightarrow{X}(g), \overleftarrow{Y}(g)) &= \overrightarrow{X}(g)L + \overleftarrow{Y}(g)L + (T_g\xi)(\overrightarrow{X}(g))L + (T_g\xi)(\overleftarrow{Y}(g))L \\ &= \overrightarrow{X}(g)L + \overleftarrow{Y}(g)L - \overrightarrow{Y}(\xi(g))L + (T_g\xi)(\overrightarrow{X}(g))L \\ &\quad +(T_g\xi)(\overleftarrow{Y}(g))L + \overrightarrow{Y}(\xi(g))L \\ &= -\Theta_L^-(g)(\overrightarrow{X}(g), \overleftarrow{Y}(g)) + \left[(\mathcal{P}^\tau\xi, \xi)^*\Theta_L^+\right](g)(\overrightarrow{X}(g), \overleftarrow{Y}(g)). \end{aligned}$$

This implies that

$$(\mathcal{P}^\tau\xi, \xi)^*\Theta_L^+ - \Theta_L^- = d(\mathcal{S}_\xi L).$$

Thus, we conclude that $(\mathcal{P}^\tau\xi, \xi)^*\Omega_L = \Omega_L$. ◇

4.5. **Lie groupoid morphisms and reduction.** Let (Φ, Φ_0) be a Lie groupoid morphism between the Lie groupoids $G \rightrightarrows M$ and $G' \rightrightarrows M'$. The prolongation $\mathcal{P}^\tau\Phi : V\beta \oplus_G V\alpha \longrightarrow V\beta' \oplus_{G'} V\alpha'$ of the morphism (Φ, Φ_0) is defined by

$$\mathcal{P}_g^\tau\Phi(V, W) = (T_g\Phi(V), T_g\Phi(W)) \tag{4.14}$$

for every $(V, W) \in V_g\beta \oplus V_g\alpha$. It is easy to see that $(\mathcal{P}^\tau\Phi, \Phi)$ is a morphism of Lie algebroids.

Theorem 4.6. *Let (Φ, Φ_0) be a morphism of Lie groupoids from $G \rightrightarrows M$ to $G' \rightrightarrows M'$. Let L and L' be discrete Lagrangian functions on G and G', respectively, related by $L = L' \circ \Phi$. Then:*

(i) *for every $(g, h) \in G_2$ and every $v \in A_{\beta(g)}G$ we have that*

$$D_{\mathrm{DEL}}L(g, h)(v) = D_{\mathrm{DEL}}L'\big(\Phi(g), \Phi(h)\big)\big(A_{\beta(g)}\Phi(v)\big). \tag{4.15}$$

(ii) $(\mathcal{P}^\tau\Phi, \Phi)^*\Theta_{L'}^+ = \Theta_L^+$,
(iii) $(\mathcal{P}^\tau\Phi, \Phi)^*\Theta_{L'}^- = \Theta_L^-$,
(iv) $(\mathcal{P}^\tau\Phi, \Phi)^*\Omega_{L'} = \Omega_L$.

Proof. To prove the first we notice that, if (Φ, Φ_0) is a morphism of Lie groupoids, then we have that $\Phi \circ l_g = l_{\Phi(g)} \circ \Phi$ and $\Phi \circ r_h = r_{\Phi(h)} \circ \Phi$, from where we get

$$\begin{aligned} D_{\mathrm{DEL}}L(g, h)(v) &= Tl_g(v)L + Tr_h(Ti(v))L \\ &= Tl_g(v)(L' \circ \Phi) + Tr_h(Ti(v))(L' \circ \Phi) \\ &= T\Phi(Tl_g(v))L' + T\Phi(Tr_h(Ti(v)))L' \\ &= Tl_{\Phi(g)}(T\Phi(v))L' + Tr_{\Phi(h)}(T\Phi(Ti(v)))L' \\ &= Tl_{\Phi(g)}(T\Phi(v))L' + Tr_{\Phi(h)}(Ti'(T\Phi(v)))L' \\ &= D_{\mathrm{DEL}}L'\big(\Phi(g), \Phi(h)\big)\big(A_{\beta(g)}\Phi(v)\big), \end{aligned}$$

where we have also used that $i' \circ \Phi = \Phi \circ i$ and $A_{\beta(g)}\Phi(v) = T\Phi(v)$.

For the proof of the second, we have that

$$\begin{aligned}\langle(\mathcal{P}^\tau\Phi,\Phi)^*\Theta^+_{L'}(g),(V,W)\rangle &= \langle\Theta^+_{L'}(\Phi(g)),(T_g\Phi(V),T_g\Phi(W))\rangle\\ &= ((T_g\Phi)(W))L' = WL = \langle\Theta^+_L(g),(V,W)\rangle,\end{aligned}$$

for every $(V,W)\in\mathcal{P}^\tau_gG$. The proof of the third is similar to the second, and finally, for the proof of (iv) we just take the differential in (ii). □

As an immediate consequence of the above theorem we have that

Corollary 4.7. *Let (Φ,Φ_0) be a morphism of Lie groupoids from $G\rightrightarrows M$ to $G'\rightrightarrows M'$ and suppose that $(g,h)\in G_2$.*

(i) *If $(\Phi(g),\Phi(h))$ is a solution of the discrete Euler-Lagrange equations for $L'=L\circ\Phi$, then (g,h) is a solution of the discrete Euler-Lagrange equations for L.*
(ii) *If Φ is a submersion then (g,h) is a solution of the discrete Euler-Lagrange equations for L if and only if $(\Phi(g),\Phi(h))$ is a solution of the discrete Euler-Lagrange equations for L'.*
(iii) *If Φ is an immersion, then (g,h) is a solution of the discrete Euler-Lagrange equations for L if and only if $D_{\mathrm{DEL}}L(\Phi(g),\Phi(h))$ vanishes over $\mathrm{Im}(A_\beta(g)\Phi)$.*

The case when Φ is an inmersion may be useful to modelize holonomic mechanics on Lie groupoids, which is an imprescindible tool for explicitely construct geometric integrators (see [8, 21]).

The particular case when Φ is a submersion is relevant for reduction (see Section 5.5 in this paper).

4.6. **Discrete Legendre transformations.** Given a Lagrangian $L:G\longrightarrow\mathbb{R}$ we define, just as the standard case [21], two ***discrete Legendre transformations*** $\mathbb{F}^-L:G\longrightarrow A^*G$ and $\mathbb{F}^+L:G\longrightarrow A^*G$ as follows

$$\begin{aligned}(\mathbb{F}^-L)(h)(v_{\epsilon(\alpha(h))}) &= -v_{\epsilon(\alpha(h))}(L\circ r_h\circ i), \quad \text{for } v_{\epsilon(\alpha(h))}\in A_{\alpha(h)}G,\\ (\mathbb{F}^+L)(g)(v_{\epsilon(\beta(g))}) &= v_{\epsilon(\beta(g))}(L\circ l_g), \ \text{for } v_{\epsilon(\beta(g))}\in A_{\beta(g)}G.\end{aligned}$$

Remark 4.8. Note that $(\mathbb{F}^-L)(h)\in A^*_{\alpha(h)}G$. Furthermore, if U is an open subset of M such that $\alpha(h)\in U$ and $\{X_i\}$ is a local basis of $\Gamma(\tau)$ on U then

$$\mathbb{F}^-L=\overrightarrow{X_i}(L)(X^i\circ\alpha),$$

on $\alpha^{-1}(U)$, where $\{X^i\}$ is the dual basis of $\{X_i\}$. In a similar way, if V is an open subset of M such that $\beta(g)\in V$ and $\{Y_j\}$ is a local basis of $\Gamma(\tau)$ on V then

$$\mathbb{F}^+L=\overleftarrow{Y_j}(L)(Y^j\circ\beta),$$

on $\beta^{-1}(V)$. ◇

Next, we consider the prolongation $\tau^{\tau^*}:\mathcal{P}^{\tau^*}(AG)\to A^*G$ of the Lie algebroid $\tau:AG\to M$ over the fibration $\tau^*:A^*G\to M$, that is,

$$\begin{aligned}\mathcal{P}^{\tau^*}_{v^*}(AG) = &\{(v_{\tau^*(v^*)},X_{v^*})\in A_{\tau^*(v^*)}G\times T_{v^*}(A^*G)/(T_{\tau^*(v^*)}\beta)(v_{\tau^*(v^*)})\\ &= (T_{v^*}\tau^*)(X_{v^*})\}\end{aligned}$$

for $v^*\in A^*G$. Then, we may introduce the canonical section Θ of the vector bundle $(\tau^{\tau^*})^*:(\mathcal{P}^{\tau^*}AG)^*\to A^*G$ as follows:

$$\Theta(v^*)(v_{\tau^*(v^*)},X_{v^*})=v^*(v_{\tau^*(v^*)}), \tag{4.16}$$

for $v^*\in A^*G$ and $(v_{\tau^*(v^*)},X_{v^*})\in\mathcal{P}^{\tau^*}_{v^*}(AG)$. Θ is called the ***Liouville section***. Moreover, we define the ***canonical symplectic section*** Ω associated with AG by $\Omega=-d\Theta$, where d is the differential on the Lie algebroid $\tau^{\tau^*}:\mathcal{P}^{\tau^*}(AG)\to A^*G$.

It is easy to prove that Ω is nondegenerate and closed, that is, it is a symplectic section of $\mathcal{P}^{\tau^*}(AG)$ (see [14, 23]).

Now, let $\mathcal{P}^\tau\mathbb{F}^-L$ be the prolongation of $\mathbb{F}^-L$ defined by

$$\mathcal{P}^\tau\mathbb{F}^-L = (\mathrm{Id}, T\mathbb{F}^-L)\circ A(\Phi^\alpha) : \mathcal{P}^\tau G \equiv V\beta \oplus_G V\alpha \longrightarrow \mathcal{P}^{\tau^*}(AG), \tag{4.17}$$

where $A(\Phi^\alpha) : \mathcal{P}^\tau G \equiv V\beta\oplus_G V\alpha \to \mathcal{P}^\alpha(AG)$ is the Lie algebroid isomorphism (over the identity of G) defined by (3.13) and $(Id, T\mathbb{F}^-L) : \mathcal{P}^\alpha(AG) \to \mathcal{P}^{\tau^*}(AG)$ is the map given by

$$(Id, T\mathbb{F}^-L)(v_{\epsilon(\alpha(h))}, X_h) = (v_{\epsilon(\alpha(h))}, (T_h\mathbb{F}^-L)(X_h)),$$

for $(v_{\epsilon(\alpha(h))}, X_h) \in \mathcal{P}^\alpha_h(AG)$. Since the pair $((Id, T\mathbb{F}^-L), \mathbb{F}^-L)$ is a morphism between the Lie algebroids $\mathcal{P}^\alpha(AG) \to G$ and $\mathcal{P}^{\tau^*}(AG) \to A^*G$, we deduce that $(\mathcal{P}^\tau\mathbb{F}^-L, \mathbb{F}^-L)$ is also a morphism between the Lie algebroids $\mathcal{P}^\tau G \equiv V\beta\oplus_G V\alpha \to G$ and $\mathcal{P}^{\tau^*}(AG) \to A^*G$. The following diagram illustrates the above situation:

$$\begin{tikzcd}
V\beta\oplus_G V\alpha \arrow[rr, "\mathcal{P}^\tau\mathbb{F}^-L"] \arrow[d] & & \mathcal{P}^{\tau^*}(AG) \arrow[d] \\
G \arrow[rr, "\mathbb{F}^-L"] \arrow[dr, "\alpha"'] & & A^*G \arrow[dl, "\tau^*"] \\
& M &
\end{tikzcd}$$

The prolongation $\mathcal{P}^\tau\mathbb{F}^-L$ can be explicitly written as

$$\mathcal{P}^\tau_h\mathbb{F}^-L(X_h, Y_h) = (T_h(i\circ r_{h^{-1}})(X_h), (T_h\mathbb{F}^-L)(X_h) + (T_h\mathbb{F}^-L)(Y_h)), \tag{4.18}$$

for $h\in G$ and $(X_h, Y_h) \in V_h\beta \oplus V_h\alpha$.

Proposition 4.9. *If Θ is the Liouville section of the vector bundle $(\mathcal{P}^{\tau^*}(AG))^* \to A^*G$ and $\Omega = -d\Theta$ is the canonical symplectic section of $\wedge^2(\mathcal{P}^{\tau^*}(AG))^* \to A^*G$ then*

$$(\mathcal{P}^\tau(\mathbb{F}^-L), \mathbb{F}^-L)^*\Theta = \Theta^-_L, \qquad (\mathcal{P}^\tau(\mathbb{F}^-L), \mathbb{F}^-L)^*\Omega = \Omega_L.$$

Proof. Let Θ^α_L be the section of $(\tau^\alpha)^* : \mathcal{P}^\alpha(AG)^* \to G$ defined by (4.4). Then, from (4.5) and (4.16), we deduce that

$$((Id, T\mathbb{F}^-L), \mathbb{F}^-L)^*\Theta = \Theta^\alpha_L.$$

Thus, using (4.4), we obtain that

$$(\mathcal{P}^\tau\mathbb{F}^-L, \mathbb{F}^-L)^*\Theta = \Theta^-_L.$$

Therefore, since the pair $(\mathcal{P}^\tau\mathbb{F}^-L, \mathbb{F}^-L)$ is a Lie algebroid morphism, it follows that

$$(\mathcal{P}^\tau\mathbb{F}^-L, \mathbb{F}^-L)^*\Omega = \Omega_L.$$

□

Now, we consider the prolongation $\mathcal{P}^\tau\mathbb{F}^+L$ of $\mathbb{F}^+L$ defined by

$$\mathcal{P}^\tau\mathbb{F}^+L = (Id, T\mathbb{F}^+L)\circ A(\Phi^\beta) : \mathcal{P}^\tau G \equiv V\beta \oplus_G V\alpha \longrightarrow \mathcal{P}^{\tau^*}(AG), \tag{4.19}$$

where $A(\Phi^\beta) : \mathcal{P}^\tau G \equiv V\beta\oplus_G V\alpha \to \mathcal{P}^\beta(AG)$ is the Lie algebroid isomorphism (over the identity of G) defined by (3.11) and $(Id, T\mathbb{F}^+L) : \mathcal{P}^\beta(AG) \to \mathcal{P}^{\tau^*}(AG)$ is the map given by

$$(Id, T\mathbb{F}^+L)(v_{\epsilon(\beta(g))}, X_g) = (v_{\epsilon(\beta(g))}, (T_g\mathbb{F}^+L)(X_g)),$$

for $(v_{\epsilon(\beta(g))}, X_g) \in \mathcal{P}^{\beta}_g(AG)$. As above, the pair $(\mathcal{P}^{\tau}\mathbb{F}^+L, \mathbb{F}^+L)$ is a morphism between the Lie algebroids $\mathcal{P}^{\tau}G \equiv V\beta \oplus_G V\alpha \to G$ and $\mathcal{P}^{\tau^*}(AG) \to A^*G$ and the following diagram illustrates the situation

$$\begin{tikzcd}
V\beta \oplus_G V\alpha \arrow[rr, "\mathcal{P}^{\tau}\mathbb{F}^+L"] \arrow[d] & & \mathcal{P}^{\tau^*}(AG) \arrow[d] \\
G \arrow[rr, "\mathbb{F}^+L"] \arrow[dr, "\beta"'] & & A^*G \arrow[dl, "\tau^*"] \\
& M &
\end{tikzcd}$$

We also have:

Proposition 4.10. *If Θ is the Liouville section of the vector bundle $(\mathcal{P}^{\tau^*}(AG))^* \to A^*G$ and $\Omega = -d\Theta$ is the canonical symplectic section of $\wedge^2(\mathcal{P}^{\tau^*}(AG))^* \to A^*G$ then*

$$(\mathcal{P}^{\tau}(\mathbb{F}^+L), \mathbb{F}^+L)^*\Theta = \Theta^+_L, \qquad (\mathcal{P}^{\tau}(\mathbb{F}^+L), \mathbb{F}^+L)^*\Omega = \Omega_L.$$

Remark 4.11. (i) If $\xi : G \to G$ is a smooth map then ξ is a discrete Lagrangian evolution operator for L if and only if $\mathbb{F}^-L \circ \xi = \mathbb{F}^+L$.

(ii) If $(g, h) \in G_2$ we have that

$$(D_{\mathrm{DEL}}L)(g, h) = \mathbb{F}^+L(g) - \mathbb{F}^-L(h). \tag{4.20}$$

$\diamond$

4.7. **Discrete regular Lagrangians.** First of all, we will introduce the notion of a discrete regular Lagrangian.

Definition 4.12. *A Lagrangian $L : G \to \mathbb{R}$ on a Lie groupoid G is said to be regular if the Poincaré-Cartan 2-section Ω_L is symplectic on the Lie algebroid $\mathcal{P}^{\tau}G \equiv V\beta \oplus_G V\alpha \to G$.*

Next, we will obtain necessary and sufficient conditions for a discrete Lagrangian on a Lie groupoid to be regular.

Theorem 4.13. *Let $L : G \to \mathbb{R}$ be a Lagrangian function. Then:*

a) *The following conditions are equivalent:*

(i) *L is regular.*
(ii) *The Legendre transformation $\mathbb{F}^-L$ is a local diffeomorphism.*
(iii) *The Legendre transformation $\mathbb{F}^+L$ is a local diffeomorphism.*

b) *If $L : G \to \mathbb{R}$ is regular and $(g_0, h_0) \in G_2$ is a solution of the discrete Euler-Lagrange equations for L then there exist two open subsets U_0 and V_0 of G, with $g_0 \in U_0$ and $h_0 \in V_0$, and there exists a (local) discrete Lagrangian evolution operator $\xi_L : U_0 \to V_0$ such that:*

(i) *$\xi_L(g_0) = h_0$,*
(ii) *ξ_L is a diffeomorphism and*
(iii) *ξ_L is unique, that is, if U'_0 is an open subset of G, with $g_0 \in U'_0$ and $\xi'_L : U'_0 \to G$ is a (local) discrete Lagrangian evolution operator then $\xi'_{L|U_0 \cap U'_0} = \xi_{L|U_0 \cap U'_0}$.*

Proof. $\boxed{a)}$ First we will deduce the equivalence of the three conditions

$\boxed{\text{(i)}\Rightarrow\text{(ii)}}$ If $h \in G$, we need to prove that $T_h(\mathbb{F}^-L) : T_hG \longrightarrow T_{\mathbb{F}^-L(h)}A^*G$ is a linear isomorphism. Assume that there exists $Y_h \in T_hG$ such that $T_h(\mathbb{F}^-L)(Y_h) = 0$. Since $\tau^* \circ \mathbb{F}^-L = \alpha$, then $(T_h\alpha)(Y_h) = 0$, that is, $Y_h \in V_h\alpha$.

Therefore, $(0_h, Y_h) \in V_h\beta \oplus V_h\alpha$ and, from (4.18), we have that $\mathcal{P}^\tau_h(\mathbb{F}^-L)(0_h, Y_h) = 0$. Moreover, $(\mathcal{P}^\tau_h(\mathbb{F}^-L))^*\Omega(\mathbb{F}^-L(h)) = \Omega_L(h)$ and $\Omega(\mathbb{F}^-L(h))$ and $\Omega_L(h)$ are nondegenerate. Therefore, we deduce that $\mathcal{P}^\tau_h(\mathbb{F}^-L)$ is a linear isomorphism. This implies that $Y_h = 0$. This proves that $T_h(\mathbb{F}^-L) : T_hG \to T_{\mathbb{F}^-L(h)}(A^*G)$ is a linear isomorphism. In the same way we deduce $\boxed{\text{(i)}\Rightarrow\text{(iii)}}$.

$\boxed{\text{(ii)}\Rightarrow\text{(i)}}$ We will assume that $\mathbb{F}^+L$ is a local diffeomorphism, so that

$$\mathcal{P}^\tau_g\mathbb{F}^+L : \mathcal{P}^\tau_g G \equiv V_g\beta \oplus V_g\alpha \longrightarrow \mathcal{P}^{\tau^*}_{\mathbb{F}^+L(g)}(AG)$$

is a linear isomorphism, for all $g \in G$.

On the other hand, if Ω is the canonical symplectic section of the vector bundle $\wedge^2(\mathcal{P}^{\tau^*}(AG))^* \to A^*G$ then, from Proposition 4.10, we deduce that

$$(\mathcal{P}^\tau_g\mathbb{F}^+L)^*(\Omega(\mathbb{F}^+L(g))) = \Omega_L(g).$$

Thus, since $\Omega(\mathbb{F}^+L(g))$ is nondegenerate, we conclude that $\Omega_L(g)$ is also nondegenerate, for all $g \in G$. Using the same arguments we deduce $\boxed{\text{(iii)}\Rightarrow\text{(i)}}$.

$\boxed{b)}$ Using Remark 4.11, we have that

$$(\mathbb{F}^+L)(g_0) = (\mathbb{F}^-L)(h_0) = \mu_0 \in A^*G.$$

Thus, from the first part of this theorem, it follows that there exit two open subsets U_0 and V_0 of G, with $g_0 \in U_0$ and $h_0 \in V_0$, and an open subset W_0 of A^*G such that $\mu_0 \in W_0$ and

$$\mathbb{F}^+L : U_0 \to W_0, \quad \mathbb{F}^-L : V_0 \to W_0$$

are diffeomorphisms. Therefore, using Remark 4.11, we deduce that

$$\xi_L = [(\mathbb{F}^-L)^{-1} \circ (\mathbb{F}^+L)]_{|U_0} : U_0 \to V_0$$

is a (local) discrete Lagrangian evolution operator. Moreover, it is clear that $\xi_L(g_0) = h_0$ and, from the first part of this theorem, we have that ξ_L is a diffeomorphism.

Finally, if U_0' is an open subset of G, with $g_0 \in U_0'$, and $\xi_L' : U_0' \to G$ is another (local) discrete Lagrangian evolution operator then $\xi'_{L|U_0\cap U_0'} : U_0 \cap U_0' \to G$ is also a (local) discrete Lagrangian evolution operator. Consequently, using Remark 4.11, we conclude that

$$\xi'_{L|U_0\cap U_0'} = [(\mathbb{F}^-L)^{-1} \circ (\mathbb{F}^+L)]_{|U_0\cap U_0'} = \xi_{L|U_0\cap U_0'}.$$

□

Remark 4.14. Using Remark 4.3, we deduce that the Lagrangian L is regular if and only if for every $g \in G$ and every local basis $\{X_i\}$ (respectively, $\{Y_j\}$) of $\Gamma(\tau)$ on an open subset U (respectively, V) of M such that $\alpha(g) \in U$ (respectively, $\beta(g) \in V$) we have that the matrix $\overrightarrow{X_i}(\overleftarrow{Y_j}(L))$ is regular on $\alpha^{-1}(U) \cap \beta^{-1}(V)$. ⋄

Let $L : G \to \mathbb{R}$ be a regular discrete Lagrangian on G. If $f : G \to \mathbb{R}$ is a real C^∞-function on G then, using Theorem 4.13, it follows that there exists a unique $\xi_f \in \Gamma(\pi^\tau)$ such that

$$i_{\xi_f}\Omega_L = df, \tag{4.21}$$

d being the differential of the Lie algebroid $\pi^\tau : \mathcal{P}^\tau G \equiv V\beta \oplus_G V\alpha \to G$. ξ_f is called ***the Hamiltonian section associated to*** f ***with respect to*** Ω_L.

Now, one may introduce a bracket of real functions on G as follows:

$$\{\cdot,\cdot\}_L : C^\infty(G) \times C^\infty(G) \to C^\infty(G), \quad \{f, g\}_L = -\Omega_L(\xi_f, \xi_g). \tag{4.22}$$

Note that, from (4.21) and Propositions 4.9 and 4.10, we obtain that

$$(\mathcal{P}^\tau \mathbb{F}^\pm L) \circ \xi_{\bar f \circ \mathbb{F}^\pm L} = \xi_{\bar f} \circ \mathbb{F}^\pm L, \tag{4.23}$$

for $\bar f \in C^\infty(A^*G)$, where $\mathcal{P}^\tau \mathbb{F}^\pm L : \mathcal{P}^\tau G \equiv V\beta \oplus_G V\alpha \to \mathcal{P}^{\tau^*}(AG)$ is the prolongation of $\mathbb{F}^\pm L$ (see Section 4.6) and $\xi_{\bar f}$ is the Hamiltonian section associated to the real function $\bar f$ on A^*G with respect to the canonical symplectic section Ω on $\wedge^2(\mathcal{P}^{\tau^*}(AG))^* \to A^*G$, that is, $i_{\xi_{\bar f}}\Omega = d\bar f$.

On the other hand, we consider the canonical linear Poisson bracket $\{\cdot,\cdot\} : C^\infty(A^*G) \times C^\infty(A^*G) \to C^\infty(A^*G)$ on A^*G defined by (see [14])

$$\{\bar f, \bar g\} = -\Omega(\xi_{\bar f}, \xi_{\bar g}), \quad \text{for } \bar f, \bar g \in C^\infty(A^*G). \tag{4.24}$$

We have that (see [14])

$$[\![\xi_{\bar f}, \xi_{\bar g}]\!]^{\tau^{\tau^*}} = \xi_{\{\bar f, \bar g\}}.$$

Moreover, from (4.22), (4.23), (4.24) and Propositions 4.9 and 4.10, we deduce that

$$\{\bar f \circ \mathbb{F}^\pm L, \bar g \circ \mathbb{F}^\pm L\}_L = \{\bar f, \bar g\} \circ \mathbb{F}^\pm L.$$

Using the above facts, we may prove the following result.

Proposition 4.15. *Let $L : G \to \mathbb{R}$ be a regular discrete Lagrangian.*

(i) *The Hamiltonian sections with respect to Ω_L form a Lie subalgebra of the Lie algebra $(\Gamma(\pi^\tau), [\![\cdot,\cdot]\!]^{\mathcal{P}^\tau G})$.*
(ii) *The Lie groupoid G endowed with the bracket $\{\cdot,\cdot\}_L$ is a Poisson manifold, that is, $\{\cdot,\cdot\}_L$ is skew-symmetric, it is a derivation in each argument with respect to the usual product of functions and it satisfies the Jacobi identity.*
(iii) *The Legendre transformations $\mathbb{F}^\pm L : G \to A^*G$ are local Poisson isomorphisms.*

4.8. **Discrete Hamiltonian evolution operator.** Let $L : G \to \mathbb{R}$ be a regular Lagrangian and assume, without the loss of generality, that the Legendre transformations $\mathbb{F}^+L$ and $\mathbb{F}^-L$ are global diffeomorphisms. Then, $\xi_L = (\mathbb{F}^-L)^{-1} \circ (\mathbb{F}^+L)$ is the discrete Euler-Lagrange evolution operator and one may define the ***discrete Hamiltonian evolution operator***, $\tilde\xi_L : A^*G \to A^*G$, by

$$\tilde\xi_L = \mathbb{F}^+L \circ \xi_L \circ (\mathbb{F}^+L)^{-1}. \tag{4.25}$$

From Remark 4.11, we have the following alternative definitions

$$\tilde\xi_L = \mathbb{F}^-L \circ \xi_L \circ (\mathbb{F}^-L)^{-1}, \qquad \tilde\xi_L = \mathbb{F}^+L \circ (\mathbb{F}^-L)^{-1}$$

of the discrete Hamiltonian evolution operator. The following commutative diagram illustrates the situation

$$\begin{tikzcd}
 & G \arrow[rr, "\xi_L"] \arrow[dl, "\mathbb{F}^-L"'] \arrow[dr, "\mathbb{F}^+L"] & & G \arrow[dl, "\mathbb{F}^-L"'] \arrow[dr, "\mathbb{F}^+L"] & \\
A^*G \arrow[rr, "\tilde\xi_L"] & & A^*G \arrow[rr, "\tilde\xi_L"] & & A^*G
\end{tikzcd}$$

Define the prolongation $\mathcal{P}^{\tau^*}\tilde\xi_L : \mathcal{P}^{\tau^*}(AG) \to \mathcal{P}^{\tau^*}(AG)$ of $\tilde\xi_L$ by

$$\mathcal{P}^{\tau^*}\tilde\xi_L = \mathcal{P}^\tau \mathbb{F}^+L \circ \mathcal{P}^\tau \xi_L \circ (\mathcal{P}^\tau \mathbb{F}^+L)^{-1},$$

or, alternatively (see (4.10), (4.17) and (4.19)),

$$\mathcal{P}^{\tau^*}\tilde\xi_L = \mathcal{P}^\tau \mathbb{F}^+L \circ (\mathcal{P}^\tau \mathbb{F}^-L)^{-1}, \qquad \mathcal{P}^{\tau^*}\tilde\xi_L = \mathcal{P}^\tau \mathbb{F}^-L \circ \mathcal{P}^\tau \xi_L \circ (\mathcal{P}^\tau \mathbb{F}^-L)^{-1}. \tag{4.26}$$

Proposition 4.16. *If Θ is the Liouville section of the vector bundle $(\mathcal{P}^{\tau^*}(AG))^* \to A^*G$ and $\Omega = -d\Theta$ is the canonical symplectic section of $\wedge^2(\mathcal{P}^{\tau^*}(AG))^* \to A^*G$ then*

$$(\mathcal{P}^{\tau^*}\tilde{\xi}_L, \tilde{\xi}_L)^*\Theta = \Theta + d(L \circ (\mathbb{F}^- L)^{-1}), \quad (\mathcal{P}^{\tau^*}\tilde{\xi}_L, \tilde{\xi}_L)^*\Omega = \Omega.$$

Moreover, $\tilde{\xi}_L$ is a Poisson morphism for the canonical Poisson bracket on A^*G.

Proof. The result follows using (4.25), (4.26) and Theorem 4.4 and Propositions 4.9 and 4.15. □

4.9. **Noether's theorem.** Recall that classical Noether's theorem states that a continuous symmetry of a Lagrangian leads to constants of the motion. In this section, we prove a discrete version of Noether's theorem, i.e., a theorem relating invariance of the discrete Lagrangian under some transformation with the existence of constants of the motion.

Definition 4.17. *A section X of AG is said to be a Noether's symmetry of the Lagrangian L if there exists a function $f \in C^\infty(M)$ such that*

$$dL(X^{(1,1)}) = \beta^* f - \alpha^* f.$$

In this case, L is said to be quasi-invariant under X.

When $dL(X^{(1,1)}) = -\overrightarrow{X}L + \overleftarrow{X}L = 0$, we will say that L is ***invariant*** under X or that X is an ***infinitesimal symmetry*** of the discrete Lagrangian L.

Remark 4.18. The infinitesimal invariance of the Lagrangian corresponds to a finite invariance property as follows. Let Φ_s the flow of $\overleftarrow{X}$ and $\gamma(s) = \Phi_s(\epsilon(x))$ be its integral curve with $\gamma(0) = \epsilon(x)$, where $x = \beta(g)$. Then, the integral curve of $\overleftarrow{X}$ at g is $s \mapsto r_{\gamma(s)}g = g\gamma(s)$, and the integral curve of $-\overrightarrow{X}$ through $\epsilon(x)$ is $s \mapsto \gamma(s)^{-1}$. On the other hand, if $(h, h') \in G_2$ and $Y_h \in V_h\beta$, $Z_{h'} \in V_{h'}\alpha$ then

$$(T_{(h,h')}m)(Y_h, Z_{h'}) = (T_h r_{h'})(Y_h) + (T_{h'} l_h)(Z_{h'}).$$

Using the above facts, we deduce that the integral curve μ of the vector field $-\overrightarrow{X}+\overleftarrow{X}$ on G satisfying $\mu(0) = g$ is

$$\mu(s) = \gamma(s)^{-1} g \gamma(s), \text{ for all } s.$$

Thus, the invariance of the Lagrangian may be written as

$$L\big(\gamma(s)^{-1} g \gamma(s)\big) = L(g), \text{ for all } s.$$

◇

If $L : G \to \mathbb{R}$ is a regular discrete Lagrangian, by a ***constant of the motion*** we mean a function F invariant under the discrete Euler-Lagrange evolution operator ξ_L, that is, $F \circ \xi_L = F$.

Theorem 4.19 (Discrete Noether's theorem)**.** *If X is a Noether symmetry of a discrete Lagrangian L, then the function $F = \Theta_L^-(X^{(1,1)}) - \alpha^* f$ is a **constant of the motion** for the discrete dynamics defined by L.*

Proof. We first notice that $\Theta_L^-(X^{(1,1)}) = \overrightarrow{X}L$ so that the function F is $F = \overrightarrow{X}L - \alpha^* f$.

If the Lagrangian L is quasi-invariant under X and g is a point in G, then

$$-\overrightarrow{X}(g)(L) + \overleftarrow{X}(g)(L) = f(\beta(g)) - f(\alpha(g)),$$

so that

$$\overleftarrow{X}(g)(L) = \overrightarrow{X}(g)(L) + f(\beta(g)) - f(\alpha(g)).$$

We substrate $\overrightarrow{X}(\xi_L(g))(L)$ to both sides of the above expression, so that

$$\begin{aligned}\overleftarrow{X}(g)(L) - \overrightarrow{X}(\xi_L(g))(L) &= [\overrightarrow{X}(g)(L) - f(\alpha(g))] - [\overrightarrow{X}(\xi(g))(L) - f(\alpha(\xi_L(g))]\\ &= F(g) - F(\xi_L(g)),\end{aligned}$$

from where the result immediately follows using (4.9). □

Proposition 4.20. *If X is a Noether symmetry of the discrete Lagrangian L then*

$$\mathcal{L}_{X^{(1,1)}}\Theta_L^- = d(\alpha^* f). \tag{4.27}$$

Thus, if L is regular, the complete lift $X^{(1,1)}$ is a Hamiltonian section with Hamiltonian function $F = \Theta_L^-(X^{(1,1)}) - \alpha^* f$, i.e. $i_{X^{(1,1)}}\Omega_L = dF$.

Proof. Indeed, if $dL(X^{(1,1)}) = \beta^* f - \alpha^* f$ and Y is a section of AG, we have that (see Proposition 4.2),

$$\begin{aligned}(\mathcal{L}_{X^{(1,1)}}\Theta_L^-)(Y^{(1,0)}) &= -\Omega_L(X^{(1,1)}, Y^{(1,0)}) + d(i_{X^{(1,1)}}\Theta_L^-)(Y^{(1,0)})\\ &= -\overrightarrow{Y}(\overleftarrow{X}L) + \overrightarrow{Y}(\overrightarrow{X}L) = \overrightarrow{Y}(\alpha^* f - \beta^* f)\\ &= d(\alpha^* f)(Y^{(1,0)}).\end{aligned}$$

On the other hand, using (2.9) and Proposition 4.2, we deduce that

$$\begin{aligned}(\mathcal{L}_{X^{(1,1)}}\Theta_L^-)(Y^{(0,1)}) &= -\Omega_L(X^{(1,1)}, Y^{(0,1)}) + d(i_{X^{(1,1)}}\Theta_L^-)(Y^{(0,1)})\\ &= -\overrightarrow{X}(\overleftarrow{Y}L) + \overleftarrow{Y}(\overrightarrow{X}L) = [\overleftarrow{Y}, \overrightarrow{X}](L) = 0 = d(\alpha^* f)(Y^{(0,1)}).\end{aligned}$$

Thus, (4.27) holds. From (4.27), it follows that

$$i_{X^{(1,1)}}\Omega_L = -i_{X^{(1,1)}}d\Theta_L^- = di_{X^{(1,1)}}\Theta_L^- - \mathcal{L}_{X^{(1,1)}}\Theta_L^- = d[\Theta_L^-(X^{(1,1)}) - \alpha^* f] = dF,$$

which completes the proof. □

We also have

Proposition 4.21. *The vector space of Noether symmetries of the Lagrangian $L : G \to \mathbb{R}$ is a Lie subalgebra of Lie algebra $(\Gamma(\tau), [\![\cdot,\cdot]\!])$.*

Proof. Suppose that X and Y are Noether symmetries of L and that

$$dL(X^{(1,1)}) = -\overrightarrow{X}L + \overleftarrow{X}L = \beta^* f - \alpha^* f, \tag{4.28}$$

$$dL(Y^{(1,1)}) = -\overrightarrow{Y}L + \overleftarrow{Y}L = \beta^* g - \alpha^* g, \tag{4.29}$$

with $f, g \in C^\infty(M)$. Then, using (2.9), (3.2) and (3.3), we have that

$$dL([\![X,Y]\!]^{(1,1)}) = \overrightarrow{X}(\overrightarrow{Y}L) - \overrightarrow{Y}(\overrightarrow{X}L) + \overleftarrow{X}(\overleftarrow{Y}L) - \overleftarrow{Y}(\overleftarrow{X}L). \tag{4.30}$$

On the other hand, from (2.6), (2.7), (4.28) and (4.29), we deduce that

$$\begin{aligned}&\overrightarrow{X}(\overrightarrow{Y}L) = \overrightarrow{X}(\overleftarrow{Y}L) - \alpha^*(\rho(X)(g)), \qquad \overrightarrow{Y}(\overrightarrow{X}L) = \overrightarrow{Y}(\overleftarrow{X}L) - \alpha^*(\rho(Y)(f)),\\ &\overleftarrow{X}(\overleftarrow{Y}L) = \overleftarrow{X}(\overrightarrow{Y}L) + \beta^*(\rho(X)(g)), \qquad \overleftarrow{Y}(\overleftarrow{X}L) = \overleftarrow{Y}(\overrightarrow{X}L) + \beta^*(\rho(Y)(f)).\end{aligned}$$

Thus, using (2.9) and (4.30), we obtain that

$$dL([\![X,Y]\!]^{(1,1)}) = \beta^*(\rho(X)(g) - \rho(Y)(f)) - \alpha^*(\rho(X)(g) - \rho(Y)(f)).$$

Therefore, $[\![X,Y]\!]$ is a Noether symmetry of L. □

Remark 4.22. If $L : G \to \mathbb{R}$ is a regular discrete Lagrangian then, from Propositions 4.20 and 4.21, it follows that the complete lifts of Noether symmetries of L are a Lie subalgebra of the Lie algebra of Hamiltonian sections with respect to Ω_L. ◇

5. Examples

5.1. **Pair or Banal groupoid.** We consider the pair (banal) groupoid $G = M \times M$, where the structural maps are

$$\alpha(x,y) = x, \quad \beta(x,y) = y, \quad \epsilon(x) = (x,x), \quad i(x,y) = (y,x),$$
$$m((x,y),(y,z)) = (x,z).$$

We know that the Lie algebroid of G is isomorphic to the standard Lie algebroid $\tau_M : TM \to M$ and the map

$$\Psi : AG = V_{\epsilon(M)}\alpha \to TM, \quad (0_x, v_x) \in T_xM \times T_xM \to \Psi_x(0_x, v_x) = v_x, \text{ for } x \in M,$$

induces an isomorphism (over the identity of M) between AG and TM. If X is a section of $\tau_M : AG \simeq TM \to M$, that is, X is a vector field on M then $\overrightarrow{X}$ and $\overleftarrow{X}$ are the vector fields on $M \times M$ given by

$$\overrightarrow{X}(x,y) = (-X(x), 0_y) \in T_xM \times T_yM \quad \text{and} \quad \overleftarrow{X}(x,y) = (0_x, X(y)) \in T_xM \times T_yM,$$

for $(x,y) \in M \times M$. On the other hand, if $(x,y) \in M \times M$ we have that the map

$$\begin{array}{ccc} \mathcal{P}^{\tau_M}_{(x,y)}G \equiv V_{(x,y)}\beta \oplus V_{(x,y)}\alpha & \to & T_{(x,y)}(M \times M) \simeq T_xM \times T_yM, \\ ((v_x, 0_y), (0_x, v_y)) & \to & (v_x, v_y) \end{array}$$

induces an isomorphism (over the identity of $M \times M$) between the Lie algebroids $\pi^{\tau_M} : \mathcal{P}^{\tau_M}G \equiv V\beta \oplus_G V\alpha \to G = M \times M$ and $\tau_{(M\times M)} : T(M \times M) \to M \times M$.

Now, given a discrete Lagrangian $L : M \times M \to \mathbb{R}$ then the discrete Euler-lagrange equations for L are:

$$\overleftarrow{X}(x,y)(L) - \overrightarrow{X}(y,z)(L) = 0, \quad \text{for all } X \in \mathfrak{X}(M), \tag{5.1}$$

which are equivalent to the classical discrete Euler-Lagrange equations

$$D_2L(x,y) + D_1L(y,z) = 0$$

(see, for instance, [21]). The Poincaré-Cartan 1-sections Θ^-_L and Θ^+_L on $\pi^{\tau_M} : \mathcal{P}^{\tau_M}G \simeq T(M \times M) \to G = M \times M$ are the 1-forms on $M \times M$ defined by

$$\Theta^-_L(x,y)(v_x, v_y) = -v_x(L), \quad \Theta^+_L(x,y)(v_x, v_y) = v_y(L),$$

for $(x,y) \in M \times M$ and $(v_x, v_y) \in T_xM \times T_yM \simeq T_{(x,y)}(M \times M)$.

In addition, if $\xi : G = M \times M \to G = M \times M$ is a discrete Lagrangian evolution operator then the prolongation of ξ

$$\mathcal{P}^{\tau_M}\xi : \mathcal{P}^{\tau_M}G \simeq T(M \times M) \to \mathcal{P}^{\tau_M}G \simeq T(M \times M)$$

is just the tangent map to ξ and, thus, we have that

$$\xi^*\Omega_L = \Omega_L,$$

$\Omega_L = -d\Theta^-_L = -d\Theta^+_L$ being the Poincaré-Cartan 2-form on $M \times M$. The Legendre transformations $\mathbb{F}^-L : G = M \times M \to A^*G \simeq T^*M$ and $\mathbb{F}^+L : G = M \times M \to A^*G \simeq T^*M$ associated with L are the maps given by

$$\mathbb{F}^-L(x,y) = -D_1L(x,y) \in T^*_xM, \quad \mathbb{F}^+L(x,y) = D_2L(x,y) \in T^*_yM$$

for $(x,y) \in M \times M$. The Lagrangian L is regular if and only if the matrix $\left(\dfrac{\partial^2 L}{\partial x \partial y}\right)$ is regular. Finally, a Noether symmetry is a vector field X on M such that

$$D_1L(x,y)(X(x)) + D_2L(x,y)(X(y)) = f(y) - f(x),$$

for $(x,y) \in M \times M$, where $f : M \to \mathbb{R}$ is a real C^∞-function on M. If X is a Noether symmetry then

$$x \to F(x) = D_1L(x,y)(X(x)) - f(x)$$

is a constant of the motion.

In conclusion, we recover all the geometrical formulation of the classical discrete Mechanics on the discrete state space $M \times M$ (see, for instance, [21]).

5.2. **Lie groups.** We consider a Lie group G as a groupoid over one point $M = \{\mathfrak{e}\}$, the identity element of G. The structural maps are

$$\alpha(g) = \mathfrak{e}, \ \ \beta(g) = \mathfrak{e}, \ \ \epsilon(\mathfrak{e}) = \mathfrak{e}, \ \ i(g) = g^{-1}, \ \ m(g,h) = gh, \quad \text{for } g, h \in G.$$

The Lie algebroid associated with G is just the Lie algebra $\mathfrak{g} = T_{\mathfrak{e}}G$ of G. Given $\xi \in \mathfrak{g}$ we have the left and right invariant vector fields:

$$\overleftarrow{\xi}(g) = (T_{\mathfrak{e}} l_g)(\xi), \quad \overrightarrow{\xi}(g) = (T_{\mathfrak{e}} r_g)(\xi), \quad \text{for } g \in G.$$

Thus, given a Lagrangian $L : G \longrightarrow \mathbb{R}$ its discrete Euler-Lagrange equations are:

$$(T_{\mathfrak{e}} l_{g_k})(\xi)(L) - (T_{\mathfrak{e}} r_{g_{k+1}})(\xi)(L) = 0, \quad \text{for all } \xi \in \mathfrak{g} \text{ and } g_k, g_{k+1} \in G,$$

or, $(l^*_{g_k} dL)(\mathfrak{e}) = (r^*_{g_{k+1}} dL)(\mathfrak{e})$. Denote by $\mu_k = (r^*_{g_k} dL)(\mathfrak{e})$ then the discrete Euler-Lagrange equations are written as

$$\mu_{k+1} = Ad^*_{g_k} \mu_k, \tag{5.2}$$

where $Ad : \ G \times \mathfrak{g} \longrightarrow \mathfrak{g}$ is the adjoint action of G on $\mathfrak{g}$. These equations are known as the ***discrete Lie-Poisson equations*** (see [1, 19, 20]).

Finally, an infinitesimal symmetry of L is an element $\xi \in \mathfrak{g}$ such that $(T_{\mathfrak{e}} l_g)(\xi)(L) = (T_{\mathfrak{e}} r_g)(\xi)(L)$, and then the associated constant of the motion is $F(g) = (T_{\mathfrak{e}} l_g)(\xi)(L) = (T_{\mathfrak{e}} r_g)(\xi)(L)$. Observe that all the Noether's symmetries are infinitesimal symmetries of L.

5.3. **Transformation or action Lie groupoid.** Let H be a Lie group and $\cdot :$ $M \times H \to M$, $(x,h) \in M \times H \mapsto xh$, a right action of H on M. As we know, H is a Lie groupoid over the identity element $\mathfrak{e}$ of H and we will denote by $\alpha, \beta, \epsilon, m$ and i the structural maps of H. If $\pi : M \to \{\mathfrak{e}\}$ is the constant map then is clear that the space

$$M\, {}_{\pi}\!\times_{\alpha} H = \{(x,h) \in M \times H / \pi(x) = \alpha(h)\}$$

is the cartesian product $G = M \times H$ and that $\cdot : M \times H \to M$ induces an action of the Lie groupoid H over the map $\pi : M \to \{\mathfrak{e}\}$ in the sense of Section 2.2 (see Example 6 in Section 2.2). Thus, we may consider the action Lie groupoid $G = M \times H$ over M with structural maps given by

$$\begin{array}{ll} \tilde{\alpha}_\pi(x,h) = x, \quad \tilde{\beta}_\pi(x,h) = xh, \quad \tilde{\epsilon}_\pi(x) = (x, \mathfrak{e}), \\ \tilde{m}_\pi((x,h),(xh,h')) = (x, hh'), \quad \tilde{i}_\pi(x,h) = (xh, h^{-1}). \end{array} \tag{5.3}$$

Now, let $\mathfrak{h} = T_{\mathfrak{e}}H$ be the Lie algebra of H and $\Phi : \mathfrak{h} \to \mathfrak{X}(M)$ the map given by

$$\Phi(\eta) = \eta_M, \quad \text{for } \eta \in \mathfrak{h},$$

where η_M is the infinitesimal generator of the action $\cdot : M \times H \to M$ corresponding to η. Then, Φ defines an action of the Lie algebroid $\mathfrak{h} \to \{\text{a point}\}$ over the projection $\pi : M \to \{\text{a point}\}$ and the corresponding action Lie algebroid $pr_1 :$ $M \times \mathfrak{h} \to M$ is just the Lie algebroid of $G = M \times H$ (see Example 6 in Section 2.2).

We have that $\Gamma(pr_1) \cong \{\tilde{\eta} : M \to \mathfrak{h} / \tilde{\eta} \text{ is smooth }\}$ and that the Lie algebroid structure $(\llbracket \cdot, \cdot \rrbracket_\Phi, \rho_\Phi)$ on $pr_1 : M \times H \to M$ is given by

$$\llbracket \tilde{\eta}, \tilde{\mu} \rrbracket_\Phi(x) = [\tilde{\eta}(x), \tilde{\mu}(x)] + (\tilde{\eta}(x))_M(x)(\tilde{\mu}) - (\tilde{\mu}(x))_M(x)(\tilde{\eta}), \quad \rho_\Phi(\tilde{\eta})(x) = (\tilde{\eta}(x))_M(x),$$

for $\tilde{\eta}, \tilde{\mu} \in \Gamma(pr_1)$ and $x \in M$. Here, $[\cdot, \cdot]$ denotes the Lie bracket of $\mathfrak{h}$.

If $(x,h) \in G = M \times H$ then the left-translation $l_{(x,h)} : \tilde{\alpha}_\pi^{-1}(xh) \to \tilde{\alpha}_\pi^{-1}(x)$ and the right-translation $r_{(x,h)} : \tilde{\beta}_\pi^{-1}(x) \to \tilde{\beta}_\pi^{-1}(xh)$ are given

$$l_{(x,h)}(xh, h') = (x, hh'), \quad r_{(x,h)}(x(h')^{-1}, h') = (x(h')^{-1}, h'h). \tag{5.4}$$

Now, if $\eta \in \mathfrak{h}$ then η defines a constant section $C_\eta : M \to \mathfrak{h}$ of $pr_1 : M \times \mathfrak{h} \to M$ and, using (2.6), (2.7), (5.3) and (5.4), we have that the left-invariant and the right-invariant vector fields $\overleftarrow{C}_\eta$ and $\overrightarrow{C}_\eta$, respectively, on $M \times H$ are defined by

$$\overrightarrow{C}_\eta(x,h) = (-\eta_M(x), \overrightarrow{\eta}(h)), \quad \overleftarrow{C}_\eta(x,h) = (0_x, \overleftarrow{\eta}(h)), \tag{5.5}$$

for $(x,h) \in G = M \times H$.

Note that if $\{\eta_i\}$ is a basis of $\mathfrak{h}$ then $\{C_{\eta_i}\}$ is a global basis of $\Gamma(pr_1)$.

Next, suppose that $L : G = M \times H \to \mathbb{R}$ is a Lagrangian function and for every $h \in H$ (resp., $x \in M$) we will denote by L_h (resp., L_x) the real function on M (resp., on H) given by $L_h(y) = L(y,h)$ (resp., $L_x(h') = L(x,h')$). Then, a composable pair $((x,h_k),(xh_k,h_{k+1})) \in G_2$ is a solution of the discrete Euler-Lagrange equations for L if

$$\overleftarrow{C}_\eta(x,h_k)(L) - \overrightarrow{C}_\eta(xh_k,h_{k+1})(L) = 0, \text{ for all } \eta \in \mathfrak{h},$$

or, in other terms (see (5.5))

$$\{(T_{\mathfrak{e}} l_{h_k})(\eta)\}(L_x) - \{(T_{\mathfrak{e}} r_{h_{k+1}})(\eta)\}(L_{xh_k}) + \eta_M(xh_k)(L_{h_{k+1}}) = 0, \text{ for all } \eta \in \mathfrak{h}.$$

As in the case of Lie groups, denote by $\mu_k(x,h_k) = d(L_x \circ r_{h_k})(\mathfrak{e})$. Then, the discrete Euler-Lagrange equations for L are written as

$$\mu_{k+1}(xh_k,h_{k+1}) = Ad^*_{h_k}\mu_k(x,h_k) + d(L_{h_{k+1}} \circ ((xh_k)\cdot))(e),$$

where $(xh_k)\cdot : H \to M$ is the map defined by

$$(xh_k)\cdot(h) = x(h_k h), \text{ for } h \in H.$$

In the particular case when M is the orbit of $a \in V$ under a representation of G on a real vector space V, the resultant equations were obtained by Bobenko and Suris, see [1, 2], and they were called the ***discrete Euler-Poincaré equations***.

Finally, an element $\xi \in \mathfrak{h}$ is an infinitesimal symmetry of L if

$$\xi_M(x)(L_h) - \overrightarrow{\xi}(h)(L_x) + \overleftarrow{\xi}(h)(L_x) = f(xh) - f(x)$$

where $f : M \longrightarrow \mathbb{R}$ is a real C^∞-function on M. The associated constant of the motion is

$$F(x,h) = -\xi_M(x)(L_h) + \overrightarrow{\xi}(h)(L_x) - f(x),$$

for $(x,h) \in M \times H$.

The heavy top. As a concrete example of a system on a transformation Lie groupoid we consider a discretization of the heavy top. In the continuous theory [22], the configuration manifold is the transformation Lie algebroid $\tau : S^2 \times \mathfrak{so}(3) \to S^2$ with Lagrangian

$$L_c(\Gamma,\Omega) = \frac{1}{2}\Omega \cdot I\Omega - mgl\Gamma \cdot \mathrm{e},$$

where $\Omega \in \mathbb{R}^3 \simeq \mathfrak{so}(3)$ is the angular velocity, Γ is the direction opposite to the gravity and e is a unit vector in the direction from the fixed point to the center of mass, all them expressed in a frame fixed to the body. The constants m, g and l are respectively the mass of the body, the strength of the gravitational acceleration and the distance from the fixed point to the center of mass. The matrix I is the inertia tensor of the body. In order to discretize this Lagrangian it is better to express it in terms of the matrices $\hat{\Omega} \in \mathfrak{so}(3)$ such that $\hat{\Omega}v = \Omega \times v$. Then

$$L_c(\Gamma,\Omega) = \frac{1}{2}\operatorname{Tr}(\hat{\Omega}\mathbb{I}\hat{\Omega}^T) - mgl\Gamma \cdot \mathrm{e}.$$

where $\mathbb{I} = \frac{1}{2}\operatorname{Tr}(I)I_3 - I$. We can define a discrete Lagrangian $L : G = S^2 \times SO(3) \to \mathbb{R}$ for the heavy top by

$$L(\Gamma_k,W_k) = -\frac{1}{h}\operatorname{Tr}(\mathbb{I}W_k) - hmgl\Gamma_k \cdot \mathrm{e}.$$

which is obtained by the rule $\hat{\Omega} = R^T\dot{R} \approx \frac{1}{h}R_k^T(R_{k+1} - R_k) = \frac{1}{h}(W_k - I_3)$, where $W_k = R_k^T R_{k+1}$.

The value of the action on an admissible variation is

$$\begin{aligned}\lambda(t) &= L(\Gamma_k, W_k e^{tK}) + L(e^{-tK}\Gamma_{k+1}, e^{-tK}W_{k+1})\\ &= -\frac{1}{h}\left[\mathrm{Tr}(\mathbb{I}W_k e^{tK}) + mglh^2\Gamma_k\cdot \mathrm{e} + \mathrm{Tr}(\mathbb{I}e^{-tK}W_{k+1}) + mglh^2(e^{-tK}\Gamma_{k+1})\cdot\mathrm{e}\right],\end{aligned}$$

where $\Gamma_{k+1} = W^T\Gamma_k$ (since the above pairs must be composable) and $K \in \mathfrak{so}(3)$ is arbitrary. Taking the derivative at $t = 0$ and after some straightforward manipulations we get the DEL equations

$$M_{k+1} - W_k^T M_k W_k + mglh^2(\widehat{\Gamma_{k+1}\times \mathrm{e}}) = 0$$

where $M = W\mathbb{I} - \mathbb{I}W^T$. Finally, in terms of the axial vector Π in $\mathbb{R}^3$ defined by $\hat{\Pi} = M$, we can write the equations in the form

$$\Pi_{k+1} = W_k^T\Pi_k + mglh^2\Gamma_{k+1}\times \mathrm{e}.$$

Remark 5.1. The above equations are to be solved as follows. From Γ_k, W_k we obtain $\Gamma_{k+1} = W_k\Gamma_k$ and Π_k from $\hat{\Pi}_k = W_k\mathbb{I} - \mathbb{I}W_k^T$. The DEL equation gives Π_{k+1} in terms of the above data. Finally we get W_{k+1} as the solution of the equation $\hat{\Pi}_{k+1} = W_{k+1}\mathbb{I} - \mathbb{I}W_{k+1}^T$, as in [26]. ◇

In the continuous theory, the section $X(\Gamma) = (\Gamma,\Gamma)$ of $S^2\times\mathfrak{so}(3)\to S^2$ is a symmetry of the Lagrangian (see [22]). We will show next that such a section is also a symmetry of the discrete Lagrangian. Indeed, it is easy to see that the left and right vector fields associated to X coincide $\overrightarrow{X} = \overleftarrow{X}$ and are both equal to

$$\overrightarrow{X}(\Gamma, W) = \Big((\Gamma, 0), (W, \hat{\Gamma}W)\Big) \in TG = TS^2\times TSO(3).$$

Thus $\rho^{\mathcal{P}^\tau G}(X^{(1,1)}) = 0$ so that X is a symmetry of the Lagrangian. In fact it is a symmetry of any discrete Lagrangian defined on $G = S^2\times SO(3)$. The associated constant of motion is

$$(\overrightarrow{X}L)(W,\Gamma) = \mathrm{Tr}(\mathbb{I}\hat{\Gamma}W) = \frac{1}{2}\mathrm{Tr}[(W\mathbb{I} - \mathbb{I}W^T)\hat{\Gamma}] = -\Pi\cdot\Gamma,$$

i.e. (minus) the angular momentum in the direction of the vector Γ.

5.4. **Atiyah or gauge groupoids.** Let $p : Q \to M$ be a principal G-bundle. A ***discrete connection*** on $p : Q \to M$ is a map $\mathcal{A}_d : Q\times Q \to G$ such that

$$\mathcal{A}_d(gq, hq') = h\mathcal{A}_d(q, q')g^{-1} \qquad \text{and} \qquad \mathcal{A}_d(q, q) = \mathfrak{e} \tag{5.6}$$

for $g, h \in G$ and $q, q' \in Q$, $\mathfrak{e}$ being the identity in the group G (see [12, 13]). We remark that a discrete principal connection may be considered as the discrete version of an standard (continuous) connection on $p : Q \to M$. In fact, if $\mathcal{A}_d : Q\times Q \to G$ is such a connection then it induces, in a natural way, a continuous connection $\mathcal{A}_c : TQ \to \mathfrak{g}$ defined by

$$\mathcal{A}_c(v_q) = (T_{(q,q)}\mathcal{A}_d)(0_q, v_q),$$

for $v_q \in T_qQ$. Moreover, if we choose a local trivialization of the principal bundle $p : Q \to M$ to be $G\times U$, where U is an open subset of M then, from (5.6), it follows that there exists a map $A : U\times U \to G$ such that

$$\mathcal{A}_d((g, x), (g', y)) = g'A(x, y)g^{-1}, \qquad \text{and} \qquad A(x, x) = \mathfrak{e},$$

for $(g, x), (g', x') \in G\times U$ (for more details, see [12, 13]).

On the other hand, using the discrete connection $\mathcal{A}_d$, one may identify the open subset $(p^{-1}(U)\times p^{-1}(U))/G \simeq ((G\times U)\times(G\times U))/G$ of the Atiyah groupoid

$(Q\times Q)/G$ with the product manifold $(U\times U)\times G$. Indeed, it is easy to prove that the map

$$((G\times U)\times (G\times U))/G \to (U\times U)\times G,$$
$$[((g,x),(g',y))] \to ((x,y),\mathcal{A}_d((e,x),(g^{-1}g',y))) = ((x,y),g^{-1}g'A(x,y)),$$

is bijective. Thus, the restriction to $((G\times U)\times(G\times U))/G$ of the Lie groupoid structure on $(Q\times Q)/G$ induces a Lie groupoid structure in $(U\times U)\times G$ with source, target and identity section given by

$$\begin{array}{ll} \alpha : (U\times U)\times G \to U; & ((x,y),g)\to x,\\ \beta : (U\times U)\times G \to U; & ((x,y),g)\to y,\\ \epsilon : U \to (U\times U)\times G; & x\to ((x,x),\mathfrak{e}), \end{array}$$

and with multiplication $m : ((U\times U)\times G)_2 \to (U\times U)\times G$ and inversion $i : (U\times U)\times G \to (U\times U)\times G$ defined by

$$\begin{array}{rcl} m(((x,y),g),((y,z),h)) &=& ((x,z), gA(x,y)^{-1}hA(y,z)^{-1}A(x,z)),\\ i((x,y),g) &=& ((y,x), A(x,y)g^{-1}A(y,x)). \end{array} \tag{5.7}$$

The fibre over the point $x\in U$ of the Lie algebroid $A((U\times U)\times G)$ may be identified with the vector space $T_xU\times\mathfrak{g}$. Thus, a section of $A((U\times U)\times G)$ is a pair $(X,\tilde{\xi})$, where X is a vector field on U and $\tilde{\xi}$ is a map from U on $\mathfrak{g}$. Note that the space $\Gamma(A((U\times U)\times G))$ is generated by sections of the form $(X,0)$ and $(0,C_\xi)$, with $X\in\mathfrak{X}(U)$, $\xi\in\mathfrak{g}$ and $C_\xi : U\to\mathfrak{g}$ being the constant map $C_\xi(x)=\xi$, for all $x\in U$. Moreover, an straightforward computation, using (5.7), proves that the vector fields $\overleftarrow{(X,0)}$, $\overrightarrow{(X,0)}$, $\overleftarrow{(0,C_\xi)}$ and $\overrightarrow{(0,C_\xi)}$ on $(U\times U)\times G$ are given by

$$\begin{aligned} \overleftarrow{(X,0)}((x,y),g) &= (0_x, X(y), (T_{A(x,y)}l_{gA(x,y)^{-1}}((T_yA_x)(X(y)))+\\ &\qquad - (Ad_{A(x,y)^{-1}}(T_yA_y)(X(y)))^l(g))),\\ \overrightarrow{(X,0)}((x,y),g) &= (-X(x), 0_y, -(T_{A(x,y)}l_{gA(x,y)^{-1}}((T_xA_y)(X(x))))+\\ &\qquad - (Ad_{A(x,y)^{-1}}(T_xA_x)(X(x)))^l(g))),\\ \overleftarrow{(0,C_\xi)}((x,y),g) &= (0_x,0_y,(Ad_{A(x,y)^{-1}}\xi)^l(g)),\\ \overrightarrow{(0,C_\xi)}((x,y),g) &= (0_x,0_y,\xi^r(g)), \end{aligned} \tag{5.8}$$

for $((x,y),g)\in (U\times U)\times G$, where $l_h : G\to G$ denotes the left-translation in G by $h\in G$, $Ad : G\times\mathfrak{g}\to\mathfrak{g}$ is the adjoint action of the Lie group G on $\mathfrak{g}$, η^l (respectively, η^r) is the left-invariant (respectively, right-invariant) vector field on G such that $\eta^l(\mathfrak{e})=\eta$ (respectively, $\eta^r(\mathfrak{e})=\eta$) and $A_x : U\to G$ and $A_y : U\to G$ are the maps defined by

$$A_x(y) = A_y(x) = A(x,y).$$

Now, suppose that $L : (Q\times Q)/G\to\mathbb{R}$ is a Lagrangian function on the Atiyah groupoid $(Q\times Q)/G$. Then, the discrete Euler-Lagrange equations for L are

$$\begin{array}{rcl} \overleftarrow{(X,0)}((x,y),g_k)(L) - \overrightarrow{(X,0)}((y,z),g_{k+1})(L) &=& 0,\\ \overleftarrow{(0,C_\xi)}((x,y),g_k)(L) - \overrightarrow{(0,C_\xi)}((y,z),g_{k+1})(L) &=& 0, \end{array}$$

with $X\in\mathfrak{X}(U)$, $\xi\in\mathfrak{g}$ and $(((x,y),g_k),((y,z),g_{k+1}))\in ((U\times U)\times G)_2$.

From (5.8), it follows that the above equations may be written as

$$\begin{aligned} &D_2L((x,y),g_k) + D_1L((y,z),g_{k+1}) + df_{AL}[x,y,g_k](y)+\\ &\qquad + df_{AL}[y,z,g_{k+1}](y) + df^1_{ALI}[x,y,g_k](y) + df^2_{ALI}[y,z,g_k+1](y) = 0, \end{aligned} \tag{5.9}$$

$$d(L_{(x,y,\,)}\circ l_{g_k}\circ I_{A(x,y)^{-1}})(\mathfrak{e}) - d(L_{(y,z,\,)}\circ r_{g_{k+1}})(\mathfrak{e}) = 0, \tag{5.10}$$

where $I_{\bar{g}} : G \to G$ denotes the interior automorphism in G of $\bar{g} \in G$, $L_{(\bar{x},\bar{y},\,)} : G \to \mathbb{R}$ is the function $L_{(\bar{x},\bar{y},\,)}(g) = L(\bar{x},\bar{y},g)$, and $f_{AL}[\bar{x},\bar{y},\bar{g}]$, $f^1_{ALI}[\bar{x},\bar{y},\bar{g}]$ and $f^2_{ALI}[\bar{x},\bar{y},\bar{g}]$ are the real functions on U given by

$$\begin{aligned}
f_{AL}[\bar{x},\bar{y},\bar{g}](y) &= L(\bar{x},\bar{y},\bar{g}A(\bar{x},\bar{y})^{-1}A(\bar{x},y)),\\
f^1_{ALI}[\bar{x},\bar{y},\bar{g}](y) &= L(\bar{x},\bar{y},\bar{g}A(\bar{x},\bar{y})^{-1}A(\bar{y},y)A(\bar{x},\bar{y})),\\
f^2_{ALI}[\bar{x},\bar{y},\bar{g}](y) &= L(\bar{x},\bar{y},\bar{g}A(\bar{x},\bar{y})^{-1}A(y,\bar{y})A(\bar{x},\bar{y})),
\end{aligned}$$

for $\bar{x},\bar{y},y \in U$ and $g \in G$. These equations may be considered as the discrete version of the Lagrange-Poincaré equations for a G-invariant continuous Lagrangian (see [5] for the local expression of the Lagrange-Poincaré equations).

Note that if $A : U \times U \to G$ is the constant map $A(x,y) = \mathfrak{e}$, for all $(x,y) \in U \times U$, or, in other words, $\mathcal{A}_d$ is the trivial connection then equations (5.9) and (5.10) may be written as

$$\begin{aligned}
&D_2L((x,y),g_k) + D_1L((y,z),g_{k+1}) = 0,\\
&\mu_{k+1}(y,z) = Ad^*_{g_k}\mu_k(x,y),
\end{aligned} \tag{5.11}$$

where

$$\mu_k(\bar{x},\bar{y}) = d(r^*_{g_k}L_{(\bar{x},\bar{y},\,)})(\mathfrak{e})$$

for $(\bar{x},\bar{y}) \in U \times U$ (compare equations (5.11) with equations (5.1) and (5.2)).

Discrete Elroy's beanie. As an example of a lagrangian system on an Atiyah groupoid, we consider a discretization of the Elroy's beanie, which is, probably, the most simple example of a dynamical system with a non-Abelian Lie group of symmetries. The continuous system consists in two planar rigid bodies attached at their centers of mass, moving freely in the plane. The configuration space is $Q = SE(2) \times S^1$ with coordinates (x,y,θ,ψ), where the three first coordinates describe the position and orientation of the center of mass of the first body and the last one the relative orientation between both bodies. The continuous system is described by a Lagrangian $L_c(x,y,\theta,\psi,\dot{x},\dot{y},\dot{\theta},\dot{\psi}) = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)+\frac{1}{2}I_1\dot{\theta}^2+\frac{1}{2}I_2(\dot{\theta}+\dot{\psi})^2 - V(\psi)$ where m denotes the mass of the system, I_1 and I_2 are the inertias of the first and the second body, respectively, and V is the potential energy. The system admits reduction by $SE(2)$ symmetry. In fact, the reduced lagrangian $l_c : TQ/SE(2) \simeq S^1 \times \mathbb{R} \times \mathfrak{se}(2) \to \mathbb{R}$ is

$$l_c(\psi,\dot{\psi},\Omega_1,\Omega_2,\Omega_3) = \frac{1}{2}m(\Omega_1^2+\Omega_2^2) + \frac{1}{2}(I_1+I_2)\Omega_3^2 + \frac{1}{2}\frac{I_1I_2}{I_1+I_2}\dot{\psi}^2 - V(\psi)$$

where $\mathfrak{se}(2)$ is the Lie algebra of $SE(2)$, $\Omega_1 = \xi_1$, $\Omega_2 = \xi_2$, $\Omega_3 = \xi_3 - \frac{I_2}{I_1+I_2}\dot{\psi}$ and (ξ_1,ξ_2,ξ_3) are the coordinates of an element of $\mathfrak{se}(2)$ with respect to the basis $e_1 = \begin{pmatrix} 0&0&1\\0&0&0\\0&0&0\end{pmatrix}$, $e_2 = \begin{pmatrix} 0&0&0\\0&0&1\\0&0&0\end{pmatrix}$ and $e_3 = \begin{pmatrix} 0&1&0\\-1&0&0\\0&0&0\end{pmatrix}$. Note that $\xi_1 = \dot{x}\cos\theta + \dot{y}\sin\theta$, $\xi_2 = -\dot{x}\sin\theta + \dot{y}\cos\theta$ and $\xi_3 = -\dot{\theta} - \dfrac{I_2}{I_1+I_2}\dot{\psi}$ (for more details, see [15, 27]).

In order to discretize this system, consider $g_k = \begin{pmatrix} \cos\theta_k & -\sin\theta_k & x_k\\ \sin\theta_k & \cos\theta_k & y_k\\ 0&0&1\end{pmatrix} \in SE(2)$. We construct the discrete connection $\mathcal{A}_d : (SE(2)\times S^1)\times(SE(2)\times S^1) \longrightarrow SE(2)$ defined by $\mathcal{A}_d((g_k,\psi_k),(g_{k+1},\psi_{k+1})) = g_{k+1}A(\psi_k,\psi_{k+1})g_k^{-1}$, where

$$A(\psi_k,\psi_{k+1}) = \begin{pmatrix} \cos(\frac{I_2}{I_1+I_2}\Delta\psi_k) & -\sin(\frac{I_2}{I_1+I_2}\Delta\psi_k) & 0\\ \sin(\frac{I_2}{I_1+I_2}\Delta\psi_k) & \cos(\frac{I_2}{I_1+I_2}\Delta\psi_k) & 0\\ 0&0&1\end{pmatrix}$$

Here, $\Delta\psi_k = \psi_{k+1} - \psi_k$. The discrete connection $\mathcal{A}_d$ precisely induces the mechanical connection associated with the $SE(2)$-invariant metric $\mathcal{G}$ on Q:

$$\mathcal{G} = m dx\otimes dx + m dy\otimes dy + (I_1+I_2)d\theta\otimes d\theta + I_2 d\theta\otimes d\psi + I_2 d\psi\otimes d\theta + I_2 d\psi\otimes d\psi$$

We remark that the continuous Lagrangian L_c is the kinetic energy associated with $\mathcal{G}$ minus the potential energy V.

Next, we consider the Atiyah groupoid $(Q\times Q)/SE(2)$. As we know, using the discrete connection $\mathcal{A}_d$, one may define a local isomorphism between the Atiyah groupoid $(Q\times Q)/SE(2)$ and the product manifold $U\times U\times SE(2)$, U being an open subset of $\mathbb{R}$. Then, as a local discretization of the reduced Lagrangian l_c, we introduce the discrete Lagrangian l_d on $U\times U\times SE(2)$ given by

$$\begin{aligned} l_d(\psi_k,\psi_{k+1},\Omega_{(1)k},\Omega_{(2)k},\Omega_{(3)k}) &= \frac{1}{2h^2}m\left[\Omega_{(1)k}^2+\Omega_{(2)k}^2\right] \\ &+\frac{(I_1+I_2)}{h^2}\left[1-\cos(\Omega_{(3)k})\right]+\frac{1}{2}\frac{I_1I_2}{I_1+I_2}\left(\frac{\Delta\psi_k}{h}\right)^2-V(\frac{\psi_k+\psi_{k+1}}{2}) \end{aligned}$$

where $\Omega_{(1)k}=\Delta x_k\cos\theta_k+\Delta y_k\sin\theta_k$, $\Omega_{(2)k}=-\Delta x_k\sin\theta_k+\Delta y_k\cos\theta_k$ and $\Omega_{(3)k}=-\Delta\theta_k-\frac{I_2}{I_1+I_2}\Delta\psi_k$.

Now, if we denote by $\bar q_k=(\psi_k,\psi_{k+1},\Omega_{(1)k},\Omega_{(2)k},\Omega_{(3)k})$ then

$$\begin{aligned} \overleftarrow{(0,C_{e_1})}\Big|_{\bar q_k} &= \cos(\Omega_{(3)k}+\tfrac{I_2}{I_1+I_2}\Delta\psi_k)\tfrac{\partial}{\partial\Omega_{(1)k}}-\sin(\Omega_{(3)k}+\tfrac{I_2}{I_1+I_2}\Delta\psi_k)\tfrac{\partial}{\partial\Omega_{(2)k}} \\ \overleftarrow{(0,C_{e_2})}\Big|_{\bar q_k} &= \sin(\Omega_{(3)k}+\tfrac{I_2}{I_1+I_2}\Delta\psi_k)\tfrac{\partial}{\partial\Omega_{(1)k}}+\cos(\Omega_{(3)k}+\tfrac{I_2}{I_1+I_2}\Delta\psi_k)\tfrac{\partial}{\partial\Omega_{(2)k}} \\ \overleftarrow{(0,C_{e_3})}\Big|_{\bar q_k} &= -\tfrac{\partial}{\partial\Omega_{(3)k}},\quad \overrightarrow{(0,C_{e_1})}\Big|_{\bar q_k}=\tfrac{\partial}{\partial\Omega_{(1)k}},\quad \overrightarrow{(0,C_{e_2})}\Big|_{\bar q_k}=\tfrac{\partial}{\partial\Omega_{(2)k}} \\ \overrightarrow{(0,C_{e_3})}\Big|_{\bar q_k} &= -\tfrac{\partial}{\partial\Omega_{(3)k}}+\Omega_{(2)k}\tfrac{\partial}{\partial\Omega_{(1)k}}-\Omega_{(1)k}\tfrac{\partial}{\partial\Omega_{(2)k}},\quad \overleftarrow{(\tfrac{\partial}{\partial\psi},0)}\Big|_{\bar q_k}=\tfrac{\partial}{\partial\psi_{k+1}}, \\ \overrightarrow{(\tfrac{\partial}{\partial\psi},0)}\Big|_{\bar q_k} &= -\tfrac{\partial}{\partial\psi_k}+\tfrac{I_2}{I_1+I_2}\Omega_{(2)k}\tfrac{\partial}{\partial\Omega_{(1)k}}-\tfrac{I_2}{I_1+I_2}\Omega_{(1)k}\tfrac{\partial}{\partial\Omega_{(2)k}} \end{aligned}$$

Thus, the reduced Discrete Euler-Lagrange equations

$$\overleftarrow{(0,C_{e_i})}\Big|_{\bar q_k}l_d-\overrightarrow{(0,C_{e_i})}\Big|_{\bar q_{k+1}}l_d=0,\quad \overleftarrow{(\frac{\partial}{\partial\psi},0)}\Big|_{\bar q_k}l_d-\overrightarrow{(\frac{\partial}{\partial\psi},0)}\Big|_{\bar q_{k+1}}l_d=0$$

are

$$\left\{\begin{array}{l} \Omega_{(1)k+1}=\Omega_{(1)k}\cos(\Omega_{(3)k}+\frac{I_2}{I_1+I_2}\Delta\psi_k)-\Omega_{(2)k}\sin(\Omega_{(3)k}+\frac{I_2}{I_1+I_2}\Delta\psi_k) \\ \Omega_{(2)k+1}=\Omega_{(1)k}\sin(\Omega_{(3)k}+\frac{I_2}{I_1+I_2}\Delta\psi_k)+\Omega_{(2)k}\cos(\Omega_{(3)k}+\frac{I_2}{I_1+I_2}\Delta\psi_k) \\ \Omega_{(3)k+1}=\Omega_{(3)k} \\ \dfrac{I_1I_2}{I_1+I_2}\dfrac{\psi_{k+2}-2\psi_{k+1}+\psi_k}{h^2}=-\dfrac{1}{2}\left(\dfrac{\partial V}{\partial\psi}(\dfrac{\psi_{k+2}+\psi_{k+1}}{2})+\dfrac{\partial V}{\partial\psi}(\dfrac{\psi_{k+1}+\psi_k}{2})\right) \end{array}\right.$$

These equations are a discretization of the corresponding reduced equations for the continuous system (see [15]). In a forthcoming paper [10], we will give a complete description of this example comparing with the continuous equations.

5.5. **Reduction of discrete Lagrangian systems.** Next, we will present some examples of Lie groupoid epimorphisms which allow to do reduction.

• Let G be a Lie group and consider the pair groupoid $G\times G$ over G. Consider also G as a groupoid over one point. Then we have that the map

$$\begin{array}{rccc} \Phi_l: & G\times G & \longrightarrow & G \\ & (g,h) & \mapsto & g^{-1}h \end{array}$$

is a Lie groupoid morphism, which is obviously a submersion. Thus, using Corollary 4.7, it follows that the discrete Euler-Lagrange equations for a left invariant discrete Lagrangian on $G\times G$ reduce to the discrete Lie-Poisson equations on G for the reduced Lagrangian. This case appears in [26] as was first noticed by [31], and also appear later in [1, 2, 19, 20].

Alternatively, one can do reduction of a right-invariant Lagrangian by using the morphism

$$\begin{array}{rccc} \Phi_r : & G\times G & \longrightarrow & G \\ & (g,h) & \mapsto & gh^{-1} \end{array}$$

• Let G be a Lie group acting on a manifold M by the left. We consider a discrete Lagrangian on $G\times G$ which depends on the variables of M as parameters $L_m(g,h)$. In general, the Lagrangian will not be invariant under the action of G, that is $L_m(g,h) \neq L_m(rg,rh)$. Nevertheless, it can happen that $L_m(rg,rh) = L_{r^{-1}m}(g,h)$. In such cases we can consider the Lie groupoid $G\times G\times M$ over $G\times M$ where accordingly one consider the elements in M as parameters. Then the Lagrangian can be considered as a function on the groupoid $G\times G\times M$ given by $L(g,h,m) \equiv L_m(g,h)$ so that the above property reads $L(rg,rh,rm) = L(g,h,m)$. Thus we define the reduction map

$$\begin{array}{rccc} \Phi : & G\times G\times M & \longrightarrow & G\times M \\ & (g,h,m) & \mapsto & (g^{-1}h, g^{-1}m) \end{array}$$

where on $G\times M$ we consider the transformation Lie groupoid defined by the right action $m\cdot g = g^{-1}m$. Since this map is a submersion, the Euler-Lagrange equations on $G\times G\times M$ reduces to the Euler-Lagrange equations on $G\times M$. This case occurs in the Lagrange top that was considered as an example in Section 5.3 (see also [2]).

• Another interesting case is that of a G-invariant Lagrangian L defined on the pair groupoid $L : Q\times Q \longrightarrow \mathbb{R}$, where $p : Q \longrightarrow M$ is a G-principal bundle. In this case we can reduce to the Atiyah gauge groupoid by means of the map

$$\begin{array}{rccc} \Phi : & Q\times Q & \longrightarrow & (Q\times Q)/G \\ & (q,q') & \mapsto & [(q,q')] \end{array}$$

Thus the discrete Euler-Lagrange equations reduce to the so called discrete Lagrange-Poincaré equations.

6. Conclusions and outlook

In this paper we have elucidated the geometrical framework for discrete Mechanics on Lie groupoids. Using as a main tool the natural Lie algebroid structure on the vector bundle $\pi^\tau : \mathcal{P}^\tau G \to G$ we have found intrinsic expressions for the discrete Euler-Lagrange equations. We introduce the Poincaré-Cartan sections, the discrete Legendre transformations and the discrete evolution operator in the Lagrangian and in the Hamiltonian formalism. The notion of regularity has been completely characterized and we prove the symplecticity of the discrete evolution operators. Moreover, we have studied the symmetries of discrete Lagrangians on Lie groupoids relating them with constants of the motion via Noether's Theorems. The applicablity of these developments has been stated in several interesting examples, in particular for the case of discrete Lagrange-Poincaré equations. In fact, the general theory of discrete symmetry reduction naturally follows from our results.

In this paper we have confined ourselves to the geometrical aspects of mechanics on Lie groupoids. In a forthcoming paper (see [10]) we will study the construction of geometric integrators for mechanical systems on Lie algebroids. We will introduce the exact discrete Lagrangian and we will discuss different discretizations of a continuous Lagrangian and its numerical implementation.

Another different aspect we will work on it in the future is to develop natural extensions of the above theories for forced systems and systems with holonomic and nonholonomic constraints.

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